This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

Stanford University Libraries
3 6105 001 364 269

510.5
M584

4/667

No. CCCXCVII.] NEW SERIES. [May, 1904.

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XXXIV.—NO. 1.

MACMILLAN AND CO.
London and Cambridge.
1904.

Price—One Shilling.

METCALFE AND CO. LIMITED, CAMBRIDGE.

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,

FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XXXIV.

[MAY 1904—APRIL 1905].

MACMILLAN AND CO.

London and Cambridge.

1905.

Digitized by Google

CONTENTS OF VOL. XXXIV.

	PAGE
Note on an integral function. By G. H. HARDY	1
Notes on some points in the integral calculus. By G. H. HARDY	3
A generalisation of Frullani's integral. By G. H. HARDY	11
Space analogues of a theorem of Hesse's. By E. J. NANSON	19
Factor-Tables.—Errata. By LT.-COL. A. CUNNINGHAM	24
Corrigenda in Lt.-Col. A. Cunningham's paper on Haupt-exponent tables	31
Pseudo-periodic functions analogous to the circular functions. By F. H. JACKSON	32
On the solution of linear difference equations by definite integrals. By H. A. WEBB	40
Notes on plane curves. By HAROLD HILTON	45
On the homogeneous linear difference equation of the second order with linear coefficients. By E. W. BARNES	52
Determination of successive high primes. By LT.-COL. CUNNINGHAM and H. J. WOODALL	72
On synthetic division. By H. J. WOODALL	90
Extension of a fundamental theorem in group theory. By G. A. MILLER	96
On the zeroes of a class of integral functions. By G. H. HARDY	97
Note on a former paper (pp. 11–18). By G. H. HARDY	102
On certain doubly infinite systems of twisted polygons. By R. A. ROBERTS	103
On the reduction of the problem of n bodies. By T. L. BENNETT	113
On certain modifications in the coefficients 1, 2, 4, 2, 4, ..., 2, 4, 1 used in Simpson's Rule, when either of the outside ordinates is a tangent to the curved boundary of the figure. By R. W. K. EDWARDS	121
A special continuant evaluated by Cayley. By THOMAS MUIR	126
Quadratic partition tables.—Errata. By LT.-COL CUNNINGHAM	132
Curves on a conicoid. By HAROLD HILTON	136
Notes on the geometry of cubic surfaces. By W. H. BLYTHE	139

80401

PAGE

On a paper-folding puzzle. By H. M. TAYLOR - - - - - - 142

Note on a proposition stated by Schlesinger. By E. CUNNINGHAM - - 144

On certain conditionally convergent multiple series connected with the elliptic functions. By G. H. HARDY - - - - - - - - - 146

On the factors of Fermat's numbers. By THOROLD GOSSET - - - 153

On the asymptotic expansion of an analytic function. By E. CUNNINGHAM 155

An extension of the theory of groups of movements. By HAROLD HILTON 157

On a perfect plane set. By W. H. YOUNG - - - - - - - - 160

On polygons inscribed in a binodal quartic and circumscribed about a conic. By R. A. ROBERTS - - - - - - - - - - - - - 161

On the plane quartic curve with a centre and the corresponding cone. By R. A. ROBERTS - - - - - - - - - - - - - 171

Determination of successive htgh primes. [Third paper] By LT.-COL. CUNNINGHAM and H. J. WOODALL - - - - - - - - - 184

MESSENGER OF MATHEMATICS.

NOTE ON AN INTEGRAL FUNCTION.

By *G. H. Hardy*, Trinity College, Cambridge.

§ 1. MM. P. Boutroux* and E. Lindelöf† have proved independently that if

$$f(x) = \prod_{2}^{\infty} \left\{1 + \frac{x}{n(\log n)^{\alpha}}\right\},$$

where $1 < \alpha < 2$, then although $f(x)$ and $f(-x)$ are of *genre* 0, $f(x) + f(-x)$ is of *genre* 1. And M. A. Wiman‡ has proved the same is true of

$$f(x) - c \quad (c \neq 0),$$

and, more generally, of $f(x) + \phi(x)$, where $\phi(x)$ is any integral function of order < 1. My object in writing this short note is to point out that, as in the case of some other functions which I have discussed recently,§ approximate expressions for the roots of

$$f(x) - c, \quad f(x) + f(-x)$$

may be written down almost immediately from the known asymptotic expression of $f(x)$. These expressions seem worth stating explicitly, as they make the nature of the discovery of MM. Boutroux and Lindelöf a good deal more tangible and intelligible.

It has been proved by M. Lindelöf‖ that if

$$-\pi < -\pi + \delta \leqq \theta \leqq \pi - \delta < \pi,$$

$$f(x) = e^{\frac{1+\epsilon(x)}{\alpha-1} x (\log x)^{-\alpha+1}},$$

* *Acta Mathematica*, t. XXVIII.
† *Acta Soc. Fennicae*, t. XXXI.
‡ *Arkiv for Mat. Ast. och Fysik*, B. 1.
§ *Proc. Lond. Math. Soc.* (N.S.), Vol. II.; *Quarterly Journal*, Vol. XXXVI.
‖ *l.c.*, p. 77.

where $\epsilon(x)$ is a function of x which tends uniformly to zero for $r=\infty$ for all values of θ in question. Thus $f(x)$ tends to ∞ with r on the right of the imaginary axis, and to 0 with $1/r$ on the left, leaving an arbitrarily small angle which includes the negative real axis out of account for the moment. And so the zeroes which lie in the part of the plane under consideration must be sought in the direction of the imaginary axis.

Now putting

$$f(x)=c, \quad x=\xi+i\eta \quad (\eta>0),$$

and proceeding as in my paper in the *Proc. L.M.S.* cited above, we find as a first approximation for the roots

$$\eta/(\log\eta)^{\alpha-1}=(\alpha-1)\,2p\pi,$$

$$\eta=(\alpha-1)\,2p\pi\,(\log p)^{\alpha-1},$$

$$\xi=-\tfrac{1}{2}\pi\,(\alpha-1)\frac{\eta}{\log\eta}$$

$$=-(\alpha-1)^2 p\pi^2\,(\log p)^{\alpha-2},$$

$$r=\quad(\alpha-1)\,2p\pi\,(\log p)^{\alpha-1}.$$

A similar result holds for $\eta<0$. Now

$$\Sigma\frac{1}{p\,(\log p)^{\alpha-1}} \quad (1<\alpha<2)$$

is divergent, and it follows that $f(x)-c$ is of *genre* 1. This conclusion holds whether or no roots of $f(x)-c$ exist in the neighbourhood of the negative real axis. One would naturally suppose that, in the case of $f(x)$ as in the case of $\dfrac{1}{\Gamma(x+1)}$, they do not, and as a matter of fact this conclusion follows readily from an inequality established by M. Boutroux.*

Similar considerations may be applied to the function

$$f(x)+f(-x),$$

and to more general functions.†

---

* *l.c.*, p. 41.

† A very interesting note on the subject of the Picard case and its extensions, by M. Boutroux, will be found in the *Comptes Rendus*, 1er Août, 1904.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy*, Trinity College, Cambridge.

XV.

On upper and lower integration.

§ 1. In a paper recently published in the *Proceedings of the London Mathematical Society*, Dr. W. H. Young proves the following theorem:

If f is any finite function, and F is the associated upper semi-continuous function, the upper integrals of f and F over any finite region are equal.

In this theorem f is any function of any finite number of variables which always lies between certain finite limits $-G$, $+G$. The function F is defined as follows.* Round any point x describe an interval x_δ or $(x-\delta, x+\delta)$; then $F(x)$ is the limit for $\delta=0$ of the upper limit of the values of $f(x)$ for points in this interval. It is easy to prove that $F(x)$ is *semi-continuous, i.e.* that its value at x is the limit for $\delta=0$ of the upper limit of its values in x_δ.

My object in writing this note is to point out that this theorem is still true in a slightly modified form which seems to me in some respects preferable to that given it by Dr. Young.†

I define the function $\bar{f}(x)$ in precisely the same manner as $F(x)$, except that in considering the values of x in x_δ the value of $f(x)$ *at x itself* is to be left out of consideration. It is evident that at any point the value of $F(x)$ is the greater of the two $f(x), \bar{f}(x)$.

Moreover $\bar{f}(x)$, like $F(x)$, is an upper semi-continuous function. For $\bar{f}(x)$ is semi-continuous if

$$\bar{f}(x) \geqq \bar{f}^2(x),$$

where $\bar{f}^2(x)$ is the function formed from $\bar{f}(x)$ as the latter is from $f(x)$, and this is easily verified.

* I use one-dimensional language throughout.

† In this form I proved the theorem some years ago, but I had never suspected that any conclusions of any interest could be deduced from it.

Then Dr. Young's theorem is still true if we substitute $\overline{f}(x)$ in the enunciation for $F(x)$.

For $\overline{f}(x)$ possesses two of the characteristics of $F(x)$:

(a) its attains its upper limit in any interval;

(b) its upper limit in any *open* interval is less than, or equal to, that of f (not necessarily equal, however, as is the case with F). And it may be proved that, if (a, A) is any finite interval,

$$\int_a^A \overline{f}(x)\,dx \leqq \overline{\int_a^A} f(x)\,dx,$$

by the same argument as is used by Dr. Young to prove the corresponding theorem for $F(x)$. But while it is obvious that

$$\int_a^A F(x)\,dx \geqq \int_a^A f(x)\,dx,$$

since $F(x) \geqq f(x)$, this is not the case with $\overline{f}(x)$.

To prove that

$$\int_a^A \overline{f}(x)\,dx \geqq \overline{\int_a^A} f(x)\,dx,$$

we observe first that if the upper limit of $\overline{f}$ in any partial interval δ_s is $\overline{G}_s$, there can be but a finite number of places in δ_s where $f > \overline{G}_s + \sigma$, however small be σ. Now choose intervals δ_s, so that

$$\int_a^A \overline{f}(x)\,dx > \Sigma \delta_s \overline{G}_s - \sigma'.$$

Subdivide δ_s into intervals $\delta_{s,i}$, so that the sum of those which contain points for which $f > \overline{G}_s + \sigma$ is $< \epsilon\delta_s$. Then

$$\overline{\int_a^A} f(x)\,dx \leqq \sum_{(s)} \sum_{(i)} \delta_{s,i} G_{s,i}$$

($G_{s,i}$ being the upper limit of f in $\delta_{s,i}$)

$$\leqq \sum_{(s)} \{\epsilon K \delta_s + (\overline{G}_s + \sigma)\,\delta_s\}$$

(K being the upper limit of *all* the values of f)

$$\leqq \sum_{(s)} \delta_s \overline{G}_s + (A-a)(\epsilon K + \sigma)$$

$$\leqq \int_a^{\overline{A}} \overline{f}(x)\,dx + (A-a)(\epsilon K + \sigma) + \sigma',$$

and so $\leqq \int_a^{\overline{A}} \overline{f}(x)\,dx$, as σ, σ', and ϵ are all arbitrarily small.

Thus
$$\int_a^{\overline{A}} \overline{f}(x)\,dx = \int_a^{\overline{A}} f(x)\,dx.$$

§ 2. If we confine ourselves to one dimension we are led to further refinements by introducing the functions

$$\underline{f}(x), \quad \overline{f}(x)$$

defined as is $f(x)$, except that the interval $(x-\delta, x+\delta)$ is replaced by $(x, x+\delta)$ or $(x-\delta, x)$ as the case may be. But the results so obtained are hardly of sufficient interest to be stated here.

§ 3. My reason for supposing that the form here given to Dr. Young's theorem may have some advantages is that singularities of f which are of no significance from the point of view of integration, but are retained in F, may disappear in $\overline{f}$. Thus, for instance, if $f=0$, except at a finite number of points at which it $=1$, f and F are identical, but $\overline{f}$ always 0. Moreover, we may define a whole series of functions

$$\overline{f}, \overline{f}^2, \overline{f}^3, \ldots,$$

and as
$$\overline{f} \geqq \overline{f}^2 \geqq \overline{f}^3 \ldots,$$

we may also define a function $\overline{f}^\omega$ by the equation

$$\overline{f}^\omega = \lim_{\nu=\infty} \overline{f}^\nu;$$

and it may be shown that if $\overline{f}^{\omega+1}$ is derived from $\overline{f}^\omega$ as $\overline{f}$ is from f, then $\overline{f}^\omega \geqq \overline{f}^{\omega+1}$, and hence we may, by the familiar process, define

$$\overline{f}^\alpha,$$

where α is any number of the first or second number class. If we consider the aggregate of functions $\bar{f}^\alpha$ at any point, it appears that there is a number β such that for $\alpha > \beta$ all the functions $\bar{f}^\alpha$ are the same. We may denote their common value by

$$\bar{f}^\Omega(x) = g(x),$$

and so define yet another function; and it may be shown that

$$g(x) = \bar{g}(x)$$

everywhere. All these functions are equivalent from the point of view of upper integration, and there is a certain gain of simplicity in replacing $f(x)$ by $g(x)$. Thus if $f(x)$ is $=0$ except at a closed set of points at which it is $=1$, $g(x)$ is $=0$ except at a perfect set; if the closed set is enumerable, $g(x)$ is always zero, and so on.

XVI.

A class of conditionally convergent infinite multiple integrals.

§ 1. In this note I propose to continue the discussion of a subject with which I have already dealt to some extent in two earlier notes. In Note X. I explained what I meant by conditionally convergent infinite multiple integrals, and showed that they are precisely analogous to conditionally convergent simple integrals, and in Note XI. I proved that some particular multiple integrals were convergent, according to the definition given in Note X. In this note I shall prove the convergence of a class of multiple integrals which is the natural generalisation of a familiar class of simple integrals. These integrals are closely analogous to a class of conditionally convergent multiple series which I have considered elsewhere (*Proc. Lond. Math. Soc.*, Ser. 2, Vol. I., p. 124).

§ 2. Let us consider the integral

$$(1) \quad \int_0^\infty \int_0^\infty \dots \int_0^\infty e^{(c_1x_1+c_2x_2+\dots+c_nx_n)i} \phi(a_1x_1+\dots+a_nx_n) \, dx_1 dx_2 \dots dx_n,$$

where $$a_1 > 0, \quad a_2 > 0, \ \ldots, \ a_n > 0,$$
$$c_1 \gtrless 0, \quad c_2 \gtrless 0, \ \ldots, \ c_n \gtrless 0,$$

and $\phi(u)$ is a positive and continuous function of u which tends steadily to 0 for $u = \infty$.

We transform the integral by the substitutions

$$|c_1| x = \xi_1, \ \ldots, \ |c_n| x = \xi_n,$$

and write α_1 for $\alpha_1 / |c_1|, \ldots$.

The integral becomes, on writing x again for ξ and neglecting a numerical factor,

$$(2) \quad \int_0^\infty \int_0^\infty \ldots \int_0^\infty \exp.\ \{(\Sigma\gamma x)\, i\}\, \phi(\Sigma\alpha x)\, dx_1 \ldots dx_n,$$

where $\gamma_1, \ldots, \gamma_n$ are each ± 1. It follows from the definitions which I have adopted that (1) and (2) converge or diverge together.

Now if D denotes the volume

$$(0, 2k_1\pi;\ 0, 2k_2\pi;\ \ldots;\ 0, 2k_n\pi),$$

and $k_1, \ldots, k_n$ are integers,

$$(3) \quad \iint_{(D)} \ldots \int \exp.\ \{(\Sigma\gamma x)\, i\}\, \phi(\Sigma\alpha x)\, dx_1 \ldots dx_n$$
$$= \sum_{(\nu)} \int_{\nu_1\pi}^{(\nu_1+1)\pi} dx_1 \ldots \int_{\nu_n\pi}^{(\nu_n+1)\pi} dx_n \left[\exp.\ \{(\Sigma\gamma x)\, i\}\, \phi(\Sigma\alpha x)\right],$$
$$(\nu_1 = 0, 1, \ldots, 2k_1 - 1;\ \ldots;\ \nu_n = 0, 1, \ldots, 2k_n - 1)$$
$$= \int_0^\pi dx_1 \int_0^\pi dx_2 \ldots \int_0^\pi X\, dx_n,$$

where $$X = \exp.\ \{(\Sigma\gamma x)\, i\} \sum_{(\nu)} \{(-)^{\Sigma\nu} \phi[\Sigma\alpha(x + \nu\pi)]\}.$$

Now if $\phi(u)$ satisfies certain further conditions, the series

$$\sum_{(\nu)} \{(-)^{\Sigma\nu} \phi[\Sigma\alpha(x + \nu\pi)]\}$$

is uniformly convergent (when $k_1, \ldots, k_n$ are made infinite) throughout the region $(0, \pi;\ 0, \pi;\ \ldots;\ 0, \pi)$.

This is very readily proved on the assumption that $\phi(u)$ has continuous derivates of at least the first n orders, and

that $\phi^{(n)}(u)$ is of constant sign, and decreases steadily in absolute value as u increases.

For consider the group of 2^n terms for which

$$\nu_1 = 2\mu_1,\ 2\mu_1 + 1;\ \ldots;\ \nu_n = 2\mu_n,\ 2\mu_n + 1.$$

Clearly their sum is numerically equal to

$$\alpha_1\alpha_2\ldots\alpha_n \int_0^\pi d\epsilon_1 \int_0^\pi d\epsilon_2 \ldots \int_0^\pi d\epsilon_n\, \phi^{(n)}\{\Sigma\alpha\,(x + 2\mu\pi + \epsilon)\}.$$

Hence (2) is numerically equal to

$$\alpha_1\alpha_2\ldots\alpha_n \int_0^\pi dx_1 \int_0^\pi dx_2 \ldots \int_0^\pi dx_n \int_0^\pi d\epsilon_1 \int_0^\pi d\epsilon_2 \ldots \int_0^\pi V d\epsilon_n,$$

where $$V = \exp.\{(\Sigma\gamma x)\,i\} \sum_{(\mu)} \phi^{(n)}\{\Sigma\alpha\,(x + 2\mu\pi + \epsilon)\},$$

the summation with respect to μ_1 extending from 0 to $k_1 - 1$, and so on.

Now the series

$$(4)\quad \sum_{(\mu)} \phi^{(n)}\{\Sigma\alpha\,(x + 2\mu\pi + \epsilon)\}$$

is uniformly convergent for all values of the x's and the ϵ's in question. For the sum of the moduli of any number of its terms is less than the sum of the corresponding terms of the series

$$\sum_{(\mu)} |\,\phi^{(n)}(\Sigma 2\alpha\mu\pi)\,|.$$

But this series converges or diverges with the integral

$$\int_0^\infty \int_0^\infty \ldots \int_0^\infty |\,\phi^{(n)}(\Sigma 2\alpha y\pi)\,|\,dy_1\ldots dy_n,$$

since

$$\sum_{\mu_1=1}^{k_1} \sum_{\mu_2=1}^{k_2} \ldots \sum_{\mu_n=1}^{k_n} |\,\phi^{(n)}(\Sigma 2\alpha\mu\pi)\,|$$

$$< \int_0^{k_1} \int_0^{k_2} \ldots \int_0^{k_n} |\,\phi^{(n)}(\Sigma 2\alpha y\pi)\,|\,dy_1\ldots dy_n,$$

$$< \sum_{\mu_1=0}^{k_1-1} \sum_{\mu_2=0}^{k_2-1} \ldots \sum_{\mu_n=0}^{k_n-1} |\,\phi^{(n)}(\Sigma 2\alpha\mu\pi)\,|.$$

And it becomes evident on working out the integrations that

$$\int_0^{k_1}\int_0^{k_2}\dots\int_0^{k_n} |\phi^{(n)}(\Sigma 2\alpha y\pi)|\, dy_1\dots dy_n$$

tends to a finite limit for $k_1, k_2, \dots, k_n = \infty$. Hence (1) is uniformly convergent, and so (3) tends to a finite limit for $k_1, k_2, \dots, k_n = \infty$; that is to say there is a quantity J such that we can, by choosing k large enough, make

$$\int_0^{2k_1\pi}\int_0^{2k_2\pi}\dots\int_0^{2k_n\pi} \exp.\{\Sigma(\gamma x)\, i\}\, \phi(\Sigma\alpha x)\, dx_1\dots dx_n$$

differ from J by less than any assigned quantity σ, for all values of $k_1, k_2, \dots, k_n > k$. It remains to remove the restriction that the upper limits should be even multiples of π.

§ 3. In order to do this we observe that

$$\int_0^{\infty}\int_0^{\infty}\dots\int_0^{\infty} |\phi^{(n)}(\Sigma 2\alpha y\pi + A)|\, dy_1\dots dy_n$$

tends to the limit 0 for $A = \infty$, and that the preceding argument shows that the same is true of

$$\sum_{(\mu)} |\phi^n(\Sigma 2\alpha\mu\pi + A)|.$$

From this it follows, by an argument similar to that of § 2, that

$$\int_0^{2k_1\pi}\int_0^{2k_2\pi}\dots\int_0^{2k_n\pi} \exp.\{\Sigma(\gamma x)\, i\}\, \phi(\Sigma\alpha x + A)\, dx_1\dots dx_n$$

tends to the limit 0 for $A = \infty$.

Now if $0 < \delta_n < 2\pi$,

$$\left| J - \int_0^{2k_1\pi}\int_0^{2k_2\pi}\dots\int_0^{2k_n\pi+\delta_n} \right|$$

$$\leqq \left| J - \int_0^{2k_1\pi}\int_0^{2k_2\pi}\dots\int_0^{2k_n\pi} \right|$$

$$+ \left| \int_0^{2k_1\pi}\int_0^{2k_2\pi}\dots\int_{2k_n\pi}^{2k_n\pi+\delta_n} \right|.$$

We can choose K so that the first term $< \frac{1}{2}\sigma$ if only

$k_1, k_2, \ldots, k_n > K$. The second is $< J'\delta_n$, where J' is the modulus of the numerically greatest value of

$$\int_0^{2k_1\pi}\int_0^{2k_2\pi}\ldots\int_0^{2k_{n-1}\pi} \exp.\{\Sigma(\gamma x)\,i\}\,\phi\,(\Sigma\alpha x+\alpha_n x_n)\,dx_1\ldots dx_{n-1},$$

(the summations applying now only to $x_1, \ldots, x_{n-1}$) for values of x_n in $(2k_n\pi,\ 2k_n\pi+\delta_n)$. We can choose K_1 so that

$$J' < \frac{\sigma}{4\pi} \quad (k_n > K_1),$$

and then

$$\left| J - \int_0^{2k_1\pi}\int_0^{2k_2\pi}\ldots\int_0^{2k_n\pi+\delta_n} \right| < \sigma,$$

if all the k's are greater than the greater of K and K_1.

Thus we have removed the restriction that the upper limit of integration with respect to x_n should be a multiple of 2π. It is clear that by a repetition of this argument the restriction can be removed from all the upper limits.

Thus, *if* $a_1, a_2, \ldots, a_n > 0$, *and* $c_1, c_2, \ldots, c_n \geqq 0$, *and* $\phi(u)$ *is a function of* u *which tends steadily to* 0 *for* $u=\infty$, *and has an* n^{th} *derivate which is continuous and tends steadily to* 0 *for* $u=\infty$, *the integral*

$$(1)\quad \int_0^\infty\int_0^\infty\ldots\int_0^\infty e^{(c_1x_1+\ldots+c_nx_n)i}\,\phi\,(a_1x_1+a_2x_2+\ldots+a_nx_n)\,dx_1\ldots dx_n$$

is convergent.

If, *e.g.*
$$\phi(u) = u^{-s} \quad (0<s<n),$$

$$\phi^{(n)}(u) = (-)^n\,s\,(s+1)\ldots(s+n-1)\,u^{-s-n},$$

so that*

$$\int_0^\infty\int_0^\infty\ldots\int_0^\infty \frac{e^{(c_1x_1+\ldots+c_nx_n)i}}{(a_1x_1+\ldots+a_nx_n)^s}\,dx_1dx_2\ldots dx_n$$

is convergent.

It is evident that the condition that $\phi^{(n)}(u)$ tends steadily to 0 need only be satisfied for values of u greater than some fixed value. We may, *e.g.*, suppose

$$\phi(u) = \frac{1}{u^2+k^2}.$$

* It is easy to see that the infinity at the origin does not affect the truth of this provided $s<n$.

A GENERALISATION OF FRULLANI'S INTEGRAL.

By *G. H. Hardy*, Trinity College, Cambridge.

§ 1. In this paper I propose to investigate a general formula for the integral

$$(1)\quad \int_0^\infty \frac{A\phi(ax)+B\phi(bx)+\ldots+K\phi(kx)}{x^m}\,dx,$$

in which ϕ is an arbitrary function subject to certain restrictions, a, b, ..., k are positive, m is a positive integer, and A, B, ..., K constants so chosen that

$$\Sigma A a^p = 0,$$

for $p = 0, 1, \ldots, m-1$. These last conditions are, with the restrictions which will be imposed on ϕ, sufficient, and, *in general*, necessary, to ensure that the subject of integration does not become infinite for $x = 0$.

I am not aware that the general formula has been given before, though special cases have. Thus the results for $\phi(x) = \cos x$ appear in Wolstenholme's *Problems*.* In these cases the integral enters into a class considered by Leslie Ellis† and Dr. Glaisher.‡ A further generalisation for the case in which $\phi(x) = e^{-ax}$ has been given by Hurwitz§ in a paper to which I shall return later. The method adopted here is more elementary than that of any of these authors.

§ 2. We must distinguish two cases.

(*a*) *Case* (i): $m = 1$. The argument is in this case practically the same as in the ordinary proof of Frullani's formula

$$(2)\quad \int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}\,dx = \{\phi(\infty)-\phi(0)\}\log(a/b).$$

---

* *l.c.* (3rd Ed.), p. 339.

† 'Evaluation of certain definite integrals,' *Camb. Math. Journal*, III., p. 185.

‡ 'On the evaluation of a class of definite integrals,' *Proc. Lond. Math. Soc.*, IV., p. 191.

§ 'Ueber die Anwendung eines functionen-theoretischen Principes auf gewisse bestimmte Integrale,' *Math. Ann.*, LIII., p. 220.

We suppose first that $\phi(x)$ is continuous for all positive values of x, 0 and ∞ included, and that $a \geqq b \geqq c \ldots \geqq k$.

Then

$$(3)\quad \int_h^H \frac{\Sigma A \phi(ax)}{x}\,dx = \Sigma A \int_{ah}^{aH} \frac{\phi(u)}{u}\,du$$

$$= \left[\left\{(\Sigma A)\int_{ah}^{kH} + A\int_{kH}^{aH} + B\left(\int_{kH}^{bH} + \int_{bh}^{ah}\right) + \ldots + K\int_{kh}^{ah}\right\}\frac{\phi u}{u}\,du\right].$$

The first term vanishes. Also we can choose H so large that if $u \leqq kH$,

$$\phi(u) = \phi(\infty) + \rho,$$

$$|\rho| < \sigma,$$

where σ is an assigned small quantity. Then

$$\int_{kH}^{aH} \frac{\phi(u)}{u}\,du = \phi(\infty)\log\left(\frac{a}{k}\right) + R,$$

where $$|R| < \sigma\int_{kH}^{aH}\frac{du}{u} < \sigma\log\frac{a}{k}.$$

A similar argument can be applied to each of the other integrals; and it follows that we can choose H so large and h so small, that (3) differs by as little as we please from

$$\phi(\infty)\left\{A\log\left(\frac{a}{k}\right) + B\log\left(\frac{b}{k}\right) + \ldots + J\log\left(\frac{j}{k}\right)\right\}$$

$$+ \phi(0)\left\{B\log\left(\frac{a}{b}\right) + C\log\left(\frac{a}{c}\right) + \ldots + K\log\left(\frac{a}{k}\right)\right\}$$

$$= \{\phi(\infty) - \phi(0)\}\,\Sigma A\log a,$$

since $\Sigma A = 0$. Thus

$$(4)\quad \int_0^\infty \frac{\Sigma A\phi(ax)}{x}\,dx = \{\phi(\infty) - \phi(0)\}\,\Sigma A\log a.$$

It is easy to see that if $\phi(\infty)$ is not determinate, but

$$\int^\infty \frac{\phi(u)}{u}\,du$$

is convergent, we must replace $\phi(\infty)$ by 0 in this formula.

§ 3. (*b*) *Case* (ii): $m>1$. In this case we suppose ϕ continuous for all finite values of x, and expansible in a Taylor's series valid near 0, and

$$\int^{\infty} \frac{\phi(u)}{u^m}\,du$$

convergent. Then

$$(5)\quad \int_h^H \frac{\Sigma A\phi(ax)}{x^m}\,dx = \Sigma A a^{m-1}\int_{ah}^{aH}\frac{\phi(u)}{u^m}\,du$$

$$= \left[\left\{(\Sigma A a^{m-1})\int_{ah}^{kH} + Aa^{m-1}\int_{kH}^{aH} + Bb^{m-1}\left(\int_{kH}^{bH}+\int_{bh}^{ah}\right)+\dots \right.\right.$$

$$\left.\left. + Kk^{m-1}\int_{kh}^{ah}\right\}\frac{\phi(u)}{u^m}\,du\right].$$

Now it is evident that

$$\lim_{H=\infty}\int_{kH}^{aH}\frac{\phi(u)}{u^m}\,du = 0,$$

and similarly for all the integrals which involve H. And

$$\int_{bh}^{ah}\frac{\phi(u)}{u^m}\,du = \phi(0)\int_{bh}^{ah}\frac{du}{u^m} + \frac{\phi'(0)}{1!}\int_{bh}^{ah}\frac{du}{u^{m-1}}+\dots$$

$$+\frac{\phi^{m-1}(0)}{m-1!}\int_{bh}^{ah}\frac{du}{u}+R,$$

where $|R|$ is less than a constant multiple of $\int_{bh}^{ah}du$ or of h. The remaining terms give

$$\frac{\phi(0)}{m-1}(b^{-m+1}-a^{-m+1})\,h^{-m+1}$$

$$+\frac{\phi'(0)}{(m-2)\,1!}(b^{-m+2}-a^{-m+2})\,h^{-m+2}+\dots$$

$$+\frac{\phi^{(m-1-\mu)}(0)}{\mu\,(m-1-\mu)!}(b^{-\mu}-a^{-\mu})\,h^{-\mu}+\dots$$

$$+\frac{\phi^{(m-1)}(0)}{(m-1)!}\log\left(\frac{a}{b}\right).$$

When we add together the various expressions ot this kind the coefficient of $h^{-\mu}$ is a multiple of

$$Bb^{m-1}(b^{-\mu}-a^{-\mu})+\dots+Kk^{m-1}(k^{-\mu}-a^{-\mu}),$$

or (since $\Sigma A a^{m-1} = 0$) of

$$\Sigma A a^{m-1-\mu} = 0.$$

The term independent of h is

$$\frac{\phi^{(m-1)}(0)}{(m-1)!}\left\{Bb^{m-1}\log\left(\frac{a}{b}\right)+\ldots+Kk^{m-1}\log\left(\frac{a}{k}\right)\right\}$$

$$= -\frac{\phi^{(m-1)}(0)}{(m-1)!}\Sigma A a^{m-1}\log a.$$

Thus

$$(6)\quad \int_0^\infty \frac{\Sigma A\phi(ax)}{x^m}\,dx = -\frac{\phi^{(m-1)}(0)}{(m-1)!}\Sigma A a^{m-1}\log a,$$

the formulæ

$$(7)\quad \int_0^\infty \frac{\Sigma A\cos ax}{x^{2n+1}}\,dx = \frac{(-)^{n-1}}{2n!}\Sigma\left(A a^{2n}\log a\right),$$

$$(8)\quad \int_0^\infty \frac{\Sigma A\sin ax}{x^{2n}}\,dx = \frac{(-)^n}{(2n-1)!}\Sigma\left(A a^{2n-1}\log a\right)$$

are given in the places already referred to. The corresponding formulæ in which m is even with cosines in the numerator, or odd with sines, are of a different type, and will be discussed later.

When $\phi(x) = e^{-x}$, (6) is a particular case of the formula

$$(9)\quad \int_0^\infty \left(\frac{Ae^{-ax}}{x^{m+1}}+\frac{Be^{-bx}}{x^{n+1}}+\ldots\right)dx$$

$$= \Sigma\frac{A(-a)^n}{n!}\left(1+\tfrac{1}{2}+\ldots+\frac{1}{n}-\gamma-\log a\right)$$

given by Hurwitz*; the integral being supposed convergent.

§ 4. Hurwitz' method may be applied to give a proof of the formula of § 3 which is interesting, though not so general or elementary as that given in § 3.

If $f(u)$ satisfies the same conditions as $\phi(u)$ in § 3, and also vanishes for $u=\infty$ more rapidly than any power of $\frac{1}{u}$, the integral

$$\int_0^\infty f(x)\,x^{s-1}\,dx$$

* There is a slight error in the formula given by Hurwitz.

is convergent if $s>0$. And it is known that it partially represents an analytic function of s which has no singularities in the s-plane save simple poles at the points

$$s=0,\ -1,\ -2,\ \ldots.$$

Now the integral

$$J=\int_0^\infty \left(\frac{f_1(x)}{x^{n_1+1}}+\ldots+\frac{f_r(x)}{x^{n_r+1}}\right)dx,$$

if convergent, is evidently the value of the analytic function

$$J(s)=\int_0^\infty \left(\Sigma\frac{f_\nu(x)}{x^{n_\nu+1}}\right)x^{s-1}dx$$

for $s=1$. And if s is real and greater than any $n_\nu+1$,

$$J(s)=\Sigma\int_0^\infty f_\nu(x)\,x^{s-n_\nu-2}\,dx$$

$$=\Sigma J_\nu(s-n_\nu-1),\ \text{say}.$$

If we consider this as an equation between a number of analytic functions, it clearly holds for *all* values of s, which are ordinary points of these functions. It follows that J is the constant term in the expansion of $\Sigma J_\nu(-n_\nu+h)$ in powers of h. The coefficients of negativepowers of h must vanish, if J is convergent. Now suppose

$$f_\nu(x)=A_\nu\phi(a_\nu x),\ \ (a_\nu>0).$$

Then if $s>0$,

$$J_\nu(s)=A_\nu\int_0^\infty \phi(a_\nu x)\,x^{s-1}dx$$

$$=\frac{(-a_\nu)^{n_\nu+1}A_\nu}{s(s+1)\ldots(s+n_\nu)}\int_0^\infty \phi^{n_\nu+1}(a_\nu x)\,x^{s+n_\nu}dx,$$

on integrating $n_\nu+1$ times by parts. The last expression gives the value of $J_\nu(s)$ for all values of s whose real part $>-n_\nu-1$. Hence

$$J_\nu(-n_\nu+h)=\frac{(-a_\nu)^{n_\nu+1}A_\nu}{(-n_\nu+h)\ldots(-1+h)\,h}\int_0^\infty \phi^{n_\nu+1}(a_\nu x)\,x^h dx.$$

Now

$$\frac{1}{(-n_\nu+h)\ldots(-1+h)\,h}=\frac{(-)^{n_\nu}}{n_\nu!}\left\{\frac{1}{h}+\left(1+\tfrac{1}{2}+\ldots+\frac{1}{n_\nu}\right)+\ldots\right\},$$

and

$$\int_0^\infty \phi^{n_\nu+1}(a_\nu x)\,x^h\,dx=-\frac{\phi^{n_\nu}(0)}{a_\nu}+h\int_0^\infty \phi^{n_\nu+1}(a_\nu x)\log x\,dx+\ldots,$$

so that the constant term is

$$\frac{a_\nu^{n_\nu}}{n_\nu!}A_\nu\phi^{n_\nu}(0)\left(1+\tfrac{1}{2}+\ldots+\frac{1}{n_\nu}\right)$$

$$-\frac{a_\nu^{n_\nu+1}}{n_\nu!}A_\nu\int_0^\infty \phi^{n_\nu+1}(a_\nu x)\log x\,dx.$$

Now suppose $n_1=n_2=\ldots=n_\nu=n$. Then, since

$$\int_0^\infty \phi^{n+1}(a_\nu x)\log x\,dx=\frac{1}{a_\nu}\int_0^\infty \phi^{n+1}(u)\{\log u-\log a_\nu\}\,du$$

$$=\frac{\log a_\nu}{a_\nu}\phi^n(0)+\frac{1}{a_\nu}\int_0^\infty \phi^{n+1}(u)\log u\,du,$$

and (since J is convergent)

$$\Sigma A_\nu a_\nu^n=0,$$

we obtain for the value of J

$$-\frac{1}{n!}\phi^n(0)\,\Sigma A_\nu a_\nu^n\log a_\nu,$$

which agrees with our former result. We have, however, introduced the unnecessary condition that

$$\lim_{u=\infty} u^n\phi(u)=0$$

for all values of n.

§ 5. There are many particular cases in which

$$\int_0^\infty \frac{\Sigma A\phi(ax)}{x^m}\,dx$$

is convergent, although the equations

$$\Sigma A a^p=0\quad(p=0,1,\ldots,m-1)$$

CONTENTS.

	PAGE
Note on an integral function. By G. H. HARDY - - - - - - -	1
Notes on some points in the integral calculus. By G. H. HARDY - -	3
A generalisation of Frullani's integral. By G. H. HARDY - - - -	11

Papers for insertion should be sent to Dr. Glaisher, Trinity College, Cambridge, or to Messrs. Metcalfe and Co. Limited, Printing Office, Trinity Street, Cambridge.

An index to vols. i—xxv has been published. It is issued gratis to subscribers, but may be obtained separately, price Two Shillings.

No. CCCXCVIII.] NEW SERIES. [June, 1904.

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XXXIV.—NO. 2.

MACMILLAN AND CO.
London and Cambridge.
1904.

Price—One Shilling.

are not all satisfied. In such cases the preceding general formulæ do not usually apply.

Suppose *e.g.* that $\phi(u)$ satisfies the same conditions as in § 3, but is in addition an *even* function of u, and that m is even, then it is only necessary for convergence that

$$\Sigma a^p = 0, \quad (p = 0, 1, \ldots, m-2).$$

The integral may be reduced as follows. As before

$$\int_0^\infty \frac{\Sigma A \phi(ax)}{x^m}\, dx = \lim. \Sigma A a^{m-1} \int_{ah}^{aH} \frac{\phi(u)}{u^m}\, du.$$

Integrating the last integral $m-1$ times by parts, we obtain

$$\left[\frac{u^{1-m}}{1-m}\phi(u) - \frac{u^{2-m}}{(1-m)(2-m)}\phi'(u) + \ldots + (-)^m \frac{u^{-1}}{(1-m)(2-m)\ldots(-1)}\phi^{m-2}(u)\right]_{ah}^{aH} = (-)^m \frac{1}{(1-m)(2-m)\ldots(-1)} \int_{ah}^{aH} \frac{\phi^{m-1}(u)}{u}\, du.$$

The limit of the last term for $h=0$, $H=\infty$ is

$$\frac{1}{(m-1)!} \int_0^\infty \frac{\phi^{m-1}(u)}{u}\, du,$$

since $\phi^{m-1}(u)$ is necessarily an odd function, and so vanishes for $u=0$.

In the integrated part we may clearly neglect the part corresponding to the upper limit. The part corresponding to the lower limit is

$$\frac{1}{(m-1)(ah)^{m-1}}\left\{\phi(0) + ah\phi'(0) + \frac{(ah)^2}{2!}\phi''(0) + \ldots\right\} + \frac{1}{(m-1)(m-2)(ah)^{m-2}}\left\{\phi'(0) + ah\phi''(0) + \frac{(ah)^2}{2!}\phi'''(0) + \ldots\right\} + \ldots + \frac{1}{(m-1)!\,(ah)}\left\{\phi^{m-2}(0) + ah\phi^{m-1}(0) + \frac{(ah)^2}{2!}\phi^m(0) + \ldots\right\}.$$

The coefficients of negative powers of must vanish, and the constant term is

$$\phi^{m-1}(0) \times \left\{\frac{1}{(m-1)(m-1)!} + \frac{1}{(m-1)(m-2)(m-2)!} + \frac{1}{(m-1)!\,1!}\right\} = 0,$$

so that

$$(10)\quad \int_0^\infty \frac{\Sigma A\phi(ax)}{x^m}\,dx = \frac{1}{(m-1)!}(\Sigma Aa^{m-1})\int_0^\infty \frac{\phi^{m-1}(u)}{u}\,du.$$

Thus, *e.g.*,

$$(11)\quad \int_0^\infty \frac{\Sigma A\cos ax}{x^{2k}}\,dx = \frac{(-)^k\frac{1}{2}\pi}{2k-1!}(\Sigma Aa^{2k-1});$$

which agrees with the result given in Wolstenholme's *Problems*. If $\Sigma Aa^{2k-1}=0$ the result agrees with § 3.

§ 6. The formula (6) may in certain cases be extended to cover the case in which the number of terms of the type $A\phi(ax)$ is infinite.

Suppose, for example, that we wish to calculate the integral

$$\int_0^\infty \psi(e^{-t})\frac{e^{-t}\,dt}{t},$$

where $\psi(u)$ is a function of u defined by a power series

$$A_0 + A_1u + A_2u^2 + \ldots,$$

which has a radius of convergence >1, and vanishes for $u=1$, so that the integral is certainly convergent. The integral is clearly

$$-\int_0^\infty \frac{dt}{t}\{A_1(e^{-t}-e^{-2t}) + A_2(e^{-t}-e^{-3t}) + \ldots\}$$

$$= -A_1\log 2 - A_2\log 3 - \ldots,$$

the integration term by term being easy to justify.

Again if

$$A_0 + 2^rA_1 + 3^rA_2 + \ldots = 0$$

for $r=0, 1, \ldots, m-1$ it is easy to prove that

$$\int_0^\infty \psi(e^{-t})\frac{e^{-t}\,dt}{t^m} = \frac{(-)^m}{(m-1)!}(2^{m-1}A_1\log 2 + 3^{m-1}A_2\log 3 + \ldots).$$

The integral on the left is the same as

$$\int_0^1 \frac{\psi(u)}{\left(\log\frac{1}{u}\right)^m}\,du.$$

SPACE ANALOGUES OF A THEOREM OF HESSE'S.

By *Prof. E. J. Nanson.*

§ 1. HESSE'S theorem that if two pairs of opposite vertices of a quadrilateral are conjugate to a conic, then the third pair are conjuate also, in which case the quadrilateral is said to be self-conjugate for the conic, has two analogues in space of three dimensions. The first of these is that if two pairs of opposite edges of a tetrahedron are conjugate to a quadric, then the third pair are conjugate also. Of this we have a special case in the well-known theorem, that if two pairs of opposite edges of a tetrahedron are at right angles, then the remaining edges are at right angles.

§ 2. The other space analogue of Hesse's theorem is due to Townsend, *Messenger*, Vol. I., p. 113, and states that if a transversal cut the faces of a tetrahedron $ABCD$ in the points A', B', C', D', and if of the four pairs A, A'; B, B'; C, C'; D, D' any three are conjugate to a quadric, then the remaining pair are conjugate also. From this theorem it at once follows: (1) that the meets of corresponding faces of two tetrahedra polar to a quadric are generators of a quadric (Chasles), (2) that the spheres on AA', BB', CC', DD' as diameters have a common radical axis (Clifford), and (3) that the middle points of AA', BB', CC', DD' are coplanar.

§ 3. In space of n dimensions, Hesse's theorem has $n-1$ analogues, and the original theorem and all its analogues follow almost intuitively from a simple law in determinants. Taking the simplest case first, let α, β, α', β' denote four lines in a plane. The condition that the points $\alpha\beta$, $\alpha'\beta'$ are conjugate to a conic is found by equating to zero the determinant obtained by bordering the discriminant of the conic with the coefficients of the lines α, β; α', β'. In Salmon's notation this determinant is denoted by $\begin{pmatrix} \alpha\beta \\ \alpha'\beta' \end{pmatrix}$. Now this determinant and others of the same type are minors of the axi-symmetric determinant which, in the same notation, is denoted by $\begin{pmatrix} \alpha\beta, & \alpha'\beta' \\ \alpha\beta, & \alpha'\beta' \end{pmatrix}$.

It follows then from Kronecker's theorem* concerning the minors of an axi-symmetric determinant, that

$$\begin{pmatrix} \alpha\beta \\ \alpha'\beta' \end{pmatrix} = \begin{pmatrix} \alpha'\beta \\ \alpha\beta' \end{pmatrix} + \begin{pmatrix} \beta'\beta \\ \alpha'\alpha \end{pmatrix} \text{ ....} (1).$$

If then any two of the three determinants in (1) vanish, then the third vanishes. Thus if any two of the pairs of points $\alpha\beta, \alpha'\beta'$; $\alpha'\beta, \alpha\beta'$; $\beta'\beta, \alpha'\alpha$ are conjugate to a conic, then the third pair are conjugate also, and this is Hesse's theorem.

§ 4. If now $\alpha, \beta, \alpha', \beta'$ are planes and the determinant which is bordered is the discriminant of a quadric the equation (1) still holds. Thus if any two of the pairs of meets $\alpha\beta, \alpha'\beta'$; $\alpha'\beta, \alpha\beta'$; $\beta'\beta, \alpha'\alpha$ are conjugate to a quadric, then the third pair are conjugate also. It is obvious that the same statement applies to four planes, and a quadric in space of three or more dimensions.

§ 5. Next let $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ be six planes in ordinary space, then, with Salmon's notation, the condition that the points $\alpha\beta\gamma, \alpha'\beta'\gamma'$ are conjugate to a quadric is

$$\begin{pmatrix} \alpha\beta\gamma \\ \alpha'\beta'\gamma' \end{pmatrix} = 0,$$

But, by Kronecker's theorem, we have the identity

$$\begin{pmatrix} \alpha\beta\gamma \\ \alpha'\beta'\gamma' \end{pmatrix} = \begin{pmatrix} \alpha'\beta\gamma \\ \alpha\beta'\gamma' \end{pmatrix} + \begin{pmatrix} \beta'\beta\gamma \\ \alpha'\alpha\gamma \end{pmatrix} + \begin{pmatrix} \gamma'\beta\gamma \\ \alpha'\beta'\alpha \end{pmatrix} \text{.....} (2).$$

Hence if any three of the four pairs of meets $\alpha\beta\gamma, \alpha'\beta'\gamma'$; $\alpha'\beta\gamma, \alpha\beta'\gamma'$; $\beta'\beta\gamma, \alpha'\alpha\gamma$; $\gamma'\beta\gamma, \alpha'\beta'\alpha$ are conjugate to a quadric, then the fourth pair are conjugate also. We thus obtain Townsend's theorem relating to the tetrahedron and the transversal.

§ 6. If now $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are six planes, that is linear loci with $n-1$ degrees of freedom, in space of n dimensions, and the determinant which is bordered is the discriminant of a quadric in space of n dimensions, then the equation (2) still holds. Hence the statement immediately following that equation is true in space of three or more dimensions.

---

* *Messenger*, Vol. XXXI., p. 140.

§ 7. Proceeding now to the Townsend analogue of Hesse's theorem in space of n dimensions, let

$$a_1, a_2, \ldots, a_n, a_1', a_2', \ldots, a_n'$$

be $2n$ planes. The condition that the points $a_1a_2\ldots a_n$, $a_1'a_2'\ldots a_n'$ are conjugate to a quadric is

$$C \equiv \begin{pmatrix} a_1, a_2 \ldots a_n \\ a_1'a_2' \ldots a_n' \end{pmatrix} = 0,$$

where the first member denotes the determinant obtained by bordering the discriminant of the quadric vertically with the coefficients of the planes $a_1, a_2, \ldots, a_n$, and horizontally with the coefficients of the planes $a_1', a_2', \ldots, a_n'$. But between determinants of this type we have the Kronecker relation

$$C = \Sigma C_r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where C_r denotes the result of interchanging a_r and a_1' in C. If, then, any n of the determinants $C, C_1, C_2, \ldots, C_n$ vanish, the remaining one vanishes. We may therefore enunciate the theorem:

If a transversal meet the $n+1$ faces of an $(n+1)$-hedron $A_1A_2\ldots A_{n+1}$ in the points $A_1', A_2', \ldots, A'_{n+1}$, and any n of the $n+1$ pairs A_r, A_r' are conjugate to a quadric, then the remaining pair are conjugate also.

§ 8. If now $a_1, a_2, \ldots, a_n, a_1', a_2', \ldots, a_n'$ are $2n$ planes in space of $n+m$ dimensions, and the determinant which is bordered is the discriminant of a quadric in that space, then the relation (3) still holds, and we may therefore enunciate the theorem:

If in space of $n+m$ dimensions $a_1, a_2, \ldots, a_n, a_1', a_2', \ldots, a_n'$ are $2n$ planes or linear loci with $n+m-1$ degrees of freedom, and if of the $n+1$ pairs of linear loci

$$\begin{array}{cc} a_1a_2\ldots a_n, & a_1'a_2'\ldots a_n', \\ a_1'a_2\ldots a_n, & a_1a_2'\ldots a_n', \\ a_1a_1'\ldots a_n, & a_2a_2'\ldots a_n', \\ \text{\&c.} & \text{\&c.} \\ a_1a_2\ldots a_1', & a_na_2'\ldots a_n' \end{array}$$

with m degrees of freedom any n are conjugate to a quadric, then the remaining pair are conjugate also.

Writing μ for $n+m$, we may give to n the values $2, 3, \ldots, \mu$ in this theorem. Thus in space of μ dimensions we obtain $\mu - 1$ analogues of Hesse's theorem.

§ 9. From the theorem in § 7 we can deduce the analogue in any space of Gauss' theorem that the middle points of the diagonals of a quadrilateral are collinear. This is that the middle points of the $n+1$ joins $A_r A_r'$ in § 7 are coplanar, and the result is found by making the quadric break up into two planes, one of which is given. The n constants which remain at our disposal then enable us to satisfy the n conditions of conjugacy. Thus if a plane cut the join $A_r A_r'$ in X_r, and X_r' is the harmonic conjugate of X_r to $A_r A_r'$, then the $n+1$ points X_r' are coplanar. Taking the given plane at infinity the result stated follows.

§ 10. From § 7 can also be deduced an analogue in any space of Clifford's result already quoted. For a singly infinite system of spheres can be found to satisfy the conditions of conjugacy in § 7. This system cuts orthogonally the spheres on the $n+1$ lines $A_r A_r'$ as diameters. The latter spheres therefore have a common radical axis. From this result that in § 9 follows.

§ 11. But there exists another way in which the Kronecker relation can be used to establish this generalisation of Clifford's theorem. And this way is interesting as it leads to $\mu - 1$ analogues in space of μ dimensions, of which one only is included in the preceding theory.

Let b_{pq} denote the cosine of the angle between the planes a_p, a_q', and border the n-line determinant $|b_{pq}|$ vertically with the perpendiculars α_p from a current point on the n planes a_p and horizontally with the perpendiculars α_q' on the n planes a_p'. Denote the resulting $(n+1)$-line determinant

$$\begin{vmatrix} (b_{pq}) & \alpha_p \\ \alpha_q' & \end{vmatrix}$$

by C. Thus the equation $C=0$ represents a quadric, and if C_r represents the result of interchanging α_r, α_1' in C, then we still, by Kronecker's theorem, have the relation (3). It

follows that the $n+1$ quadrics $C, C_1, C_2, \ldots, C_n$ have a common curve or locus of intersection with $\mu - n$ degrees of freedom.

§ 12. It remains then to find an interpretation for the equation $C=0$. Now in the case $\mu = n$ this equation represents the sphere described on the join of $a_1a_2\ldots a_n$, $a_1'a_2'\ldots a_n'$ as diameter.* We thus obtain the result of § 10.

§ 13. When $n=2$, $\mu=3$, the equation $C=0$ represents the locus of the intersection of perpendicular planes drawn through the two lines a_1a_2, $a_1'a_2'$. We thus have the theorem that the locus of the intersection of perpendicular planes drawn through two straight lines in space is a quadric, and the three quadrics thus derived from the three pairs of opposite edges of a tetrahedron have a common curve of intersection.

§ 14. In general, when $n=2$, the configurations a_1a_2, $a_1'a_2'$ are $(\mu-2)$-folds or linear loci with $\mu-2$ degrees of freedom. The equation $C=0$ represents the locus of the intersection of perpendicular planes through these configurations, and the three loci thus derived from the three pairs a_1a_2, $a_1'a_2'$; $a_1'a_2$, a_1a_2'; a_1a_1', a_2a_2' have a common intersection with $\mu-2$ degrees of freedom.

§ 15. In the general case, the configurations $a_1a_2\ldots a_n$; $a_1'a_2'\ldots a_n'$ are $(\mu-n)$-folds, and the equation $C=0$ represents the quadric locus of the intersection of a $(\mu-n+1)$-fold drawn through one of these configurations with a perpendicular plane drawn through the other configuration. And the general theorem is that the $n+1$ quadrics thus derived from the $n+1$ pairs of configurations specified in § 8 intersect in a common $(\mu-n)$-fold.

Melbourne.

---

* Clifford, *Math. Papers*, pp. 94, 108, 587.

FACTOR-TABLES.

ERRATA.

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's Coll., London.

[The thanks of the author are due to Mr. H. J. Woodall, of Southport, for verifying parts of this Paper, as stated below, and for reading the proof sheets.]

1. *Factor-Tables, Errata.* In the course of some heavy factorisations it became evident that many *Errata* unfortunately exist in several of the published Factor-Tables. These errors are very provoking as being liable to lead to consequential, and often not easily detected, errors in all calculations depending on them. After much trouble arising from this cause the author has undertaken to have a *collation* made of several of these Tables.

2. *Factor-Tables examined.* The British Association Committee Report on Mathematical Tables, by Mr. (now Dr.) J. W. L. Glaisher, contains (see *Brit. Association Report*, 1873, pp. 35–39) a list of 29 Factor-Tables published at various dates from 1659 to 1873, with (in many cases) the full titles and a brief description of each. Most of those printed before 1800 are now only of historic interest, being practically superseded by the later Tables (printed since 1800); and are, moreover, very difficult to obtain now.

The following is an Abstract of the Tables examined,* showing the author's name, date of publication, and a brief indication of the scope of each Table. For full Titles, &c., see the Appendix.

	Author.	*Date.*	*Numbers* (N).	*Limit.*
1.	*Chernac, L.*	1811;	$N \neq 2n, 3n, 5n,$	1,020,000.
2.	*Barlow, P.*	1814;	All N,	10,000.
3.	*Vega, G.*, [Ed. Hülsse],	1840-49;	$N \neq 2n, 3n, 5n,$	102,000.
4.	*Köhler, H. G.*	1848-64;	$N \neq 2n, 3n, 5n, 11n,$ & p,	21,525.
5.	*Schaller, Fr.*†	1855;	All,	10,000.
6.	*Goldberg, B.*†	1862;	$N \neq 2n, 3n, 5n,$	251,650.
7.	*Rankine, W. J. M.*	1866;	All,	256.

* This includes *all* the Tables described in the British Association Report (above quoted) as giving *complete factorisation*, published from 1800 up to 1873 (the date of the Report), except one by J. M. Solomon, published in 1827, which the author has not been able to obtain, and also includes *two* others (marked †) not mentioned therein.

† These two Tables are not mentioned in the British Association Report quoted (although published previous to the date thereof, 1873).

All these Tables have the great convenience of giving *all* the factors of all numbers (excepting, of course, those of the forms $2n$, $3n$, $5n$, $11n$, p, when stated as excluded) up to the limits stated against each.

Barlow's and *Schaller's* Tables have the great convenience of including *all* numbers under 10000, whereas numbers containing the factors 2, 3, 5 are excluded from all the more extended Tables. Primes are included among the numbers (N) in the number-column in all the Tables except *Köhler's*, and the factor-column is left *blank* opposite each prime; this renders them conspicuous and easily picked out. The primes are all omitted in the number-column in *Köhler's* Table; space is saved by their omission, but with the disadvantage that primes cannot be readily picked out from this Table. *Chernac's* Table is unique in giving *all* factors of all numbers (excluding $N=2n$, $3n$, $5n$) up to the high limit 1,020,000; it is an inconveniently bulky volume, and is expensive and difficult to obtain now.

3. *Mode of collation.* The Tables were read together* in pairs, and a list was made of all discrepancies; these discrepancies were then separately examined by two† computers (working independently), and the Erratum was thus allotted to the Table in fault. This mode of collation thus affords a *complete examination* (not really a verification) of *all* the Tables except the largest (Chernac's), which has been examined hereby only up to‡ 251650, the limit of the next most extensive Table (Goldberg's).

4. *Appendix, Errata, &c.* The Appendix is divided into seven paragraphs (numbered 1 to 7) one for each set of Tables (Nos. 1 to 7) examined. Each paragraph contains (1) the full title of the work containing the Table under review; (2) a reference to the pages on which the Table in question is contained; (3) a list of the Errata found, or of the Corrigenda required. The errors are of two kinds—

---

* By the author's assistant (Mr. R. F. Woodward).

† By the author himself, and by an assistant. Mr. H. J. Woodall, of Stockport, has also kindly verified the Corrigenda in Goldberg's Table.

‡ Chernac's Table appears to have been collated *throughout* by Burckhardt with *two* sets of MS. Tables (see p. 1 of the preface to Burckhardt's Tables); but this collation appears to have included a verification *only of the least factors* of each number.

(1) *Errata in the number-column N.* These are mostly mere typographical errors, in some cases not of a kind likely to mislead, *e.g.* when the first digit only is misprinted.

(2) *Errata in the factor-column (D).* These are the most serious, as the existence of error does not usually betray itself. In these cases it has been thought generally sufficient to print the number (*N*) itself as the Argument, and the *correct* factorisation itself as the *Corrigendum*, (without printing the Erratum, *i.e.* the incorrect factorisation, at all).

The following is an abstract of the total number of errors of each kind found in the Tables examined; here

N stands for "Number of errors in Number-column."
D stands for "Number of errors in Divisor-column."

Author.	*Limit.*	*N.*	*D.*	*Author.*	*Limit.*	*N.*	*D.*	*Author.*	*Limit.*	*N.*	*D.*
Chernac,	251650;	4,	42	*Vega,*	102000;	.	11	*Schaller,*	10000;	.	99
Barlow,	10000;	2,	38	*Köhler,*	21525;	.	2	*Goldberg,*	251650;	148,	209
								Rankine,	256;	.	2

The comparatively small number of errors in the earlier Tables is quite remarkable, especially when contrasted with the large number in the later Tables.

Appendix. Factor-Tables—Errata.

1. L. Chernac's *Cribrum Arithmeticum*, Daventriæ, 1811.

Corrigenda in column of Numbers (N).

Pages 71, 74, 83, 89, 3rd line up; *for* 933 *read* 993, Ch.

Corrigenda in column of Divisors (D); [Argument *N*).

Number (N)	Divisors (D)		Number (N)	Divisors (D)		Number (N)	Divisors (D)	
11459	7.1637	Ch	220183	421.523	Ch	545483	———	B
19697	———	C	233593	191.1223	Ch	580807	———	B
19699	———	C	253057	7.36151	Ch	614629	23.26723	Ch
21473	109.197	Ch	262223	13.23.877	Ch	631369	43.14683	Ch
26401	17.1553	Ch	264797	13.20369	Ch	637447	———	B
38963	47.829	C	278039	31.8969	Ch	643657	7.91951	Ch
39859	23.1733	B	291889	13.22453	Ch	660149	7.94307	Ch
49909	29.1721	Ch	305099	17.131.137	Ch	691141	11.83.757	Ch
65113	19.23.149	B	321043	19.61.277	Ch	730379	13.19.2957	Ch
68303	167.409	B	354029	13.113.241	B	769469	———	B
68303 to 68399	Raise each line of factors one line up	B	358231	337.1063	Ch	783661	———	B
			397091	479.829	Ch	795083	———	B
68987	149.463	C	401677	41.97.101	Ch	795089	67.11867	B
76769	7.11.997	C	405581	11.36871	Ch	795091	11.11.6571	B
106601	11^2.881	Ch	444959	449.991	Ch	884867	17.52051	Ch
111067	11.23.439	Ch	467951	11.19.2239	Ch	895781	17.23.29.79	Ch
153731	17.9043	Ch	478363	17.19.1481	Ch	931219	29.163.197	B
216151	13^2.1279	Ch	480487	7.83.827	Ch	984697	7.41.47.73	Ch
			532591	61.8731	Ch			

Note.—The Corrigenda marked Ch were discovered by Chernac himself, and are printed on p. 1021 of his Tables; those marked B were discovered by Burckhardt, and are printed in the Preface (page 1) of his Tables: these are reprinted here for the sake of completeness. The new verification extends only up to 251650 (the limit of Goldberg's Table); the Corrigenda marked C were discovered thereby.

2. P. Barlow's *New Mathematical Tables, &c.*, London, 1814. *Corrigenda* in Tab. I. of *Factors, &c.* (pages 1—167), as shown below.

(1) In "Number" column, (page 100), *for* 2973, *read* 5973.
(page 120), *for* 7048, *read* 7148.

(2) In "Factors" column, (pages 1—167); Corrigenda as below. Those marked (*) are *minor corrections*, e.g. 3^2 *for* 3.9, 3^2 *for* 9, &c. The Corrigenda in the factors of the numbers marked B were discovered by Barlow himself, and are printed on the back of his "Table of Contents"; they are reprinted here for the sake of completeness.

[The type used in these Tables is so *thin* that many figures have in the lapse of time (since 1814) become *indistinct*, so that it is likely that Errata have escaped detection.]

Number	Factors	Number	Factors	Number	Factors
465	3.5.31	3828 B	2^2.3.11.29	7322	2.7.523
1431	3^2.53*	3834	2.3^4.71*	7436	2^2.11.13^2
1698 B	2.3.283	3978 B	$2^2$3 .13.17	7668	2^2.3^2.71*
1730 B	2.5.173	4280	2^3.5.107	7795	5.1559
1917	3^2.71*	4364	2^2.1091	7894	2.3947
2140	2^2.5.107	5598	2.3^2.311*	7936	2^3.31
2504 B	2^3.313	5798	2.13.223	7964	2^2.11.181
2799	3^2.311*	5912	2^3.739	8560	2^4.5.107
2862	2.3^2.53*	6517	7^3.19	8618	2.31.139
2956	2^2.739	6660	2^2.3^2.5.37	8728	2^3.1091
3580	2^2.5.179	6786	2.3^2.13.29	9244	2^2.2311
3711 B	3.1237	6868	2^2.17.101	9275	5^2.7.53*
3718	2.11.13^2	7160	2^3.5.179		

3. C. F. Vega's *Sammlung Mathematischen Tafeln*, [Ed. *J. A. Hülsse*]; 2nd ster. Ed., Leipzig, 1849. [The first Edition by Hülsse is dated 1840].

(1) Tab. V., pp. 360—423 gives the prime factors of all numbers not divisible by 2, 3, 5 up to 102,000. Corrigenda in the factor-column of the 2nd Ed., [Argument *N*].

Number (*N*)	27293 ;	33293 ;	41779 ;	55403 ; 55517 ;
Factors of *N*	7.7.557, Ch;	13.13.197;	41.1019†;	17.3259†; 7.7.11.103†.
Number (*N*)	57103 ;	82943 ;	90983 ;	93137 ;
Factors of *N*	17.3359, Ch;	7.17.17.41, Ch;	37.2459, Ch;	11.8467, Ch;
Number (*N*)	95017;	95623.		
Factors of *N*	13.7309;	11.8693.†		

* These are corrections not of actual mistakes, but of departures from uniformity, which have been found productive of errors (by those using the Tables).
† *see* "*Note*" at foot of this paragraph, p. 28.

(2) Tab. V., pp. 424—454 gives a List of primes from 102000 to 400320.
Corrigenda in List of primes. Erase 173279, Ch; Insert 177347, Ch.
[This Table has not been examined in the present collatien].

Note.—The Errata marked Ch. were noted by Chernac (1811) as Errata in one of the *early* editions [see pp. 1021, of Chernac's Tables]. The rest were discovered in the present collation: those marked (†) are not really Errata, but are printed so indistinctly (in the copy examined) as to be liable to be misread.

4. H. G. Kohler's, *Logarithmisch Trigonometrisches Handbuch*, 9th ster. Ed., 1864, [Orig. Ed. 1848].
Tab. VIII., pp. 377—385, gives the prime factors of all numbers not divisible by 2, 3, 5, 11 up to 21525; all primes are omitted from the Number-column.
Corrigenda in the Factor-column of the 9th ster. Ed., [Argument N].

$$N = 19573 = 23.23.37; \quad N = 19579 = 7.2797.$$

5. Franz Schaller's *Primzahlen-Tafel von* 1 *bis* 10000, *oder Zerlegung aller Zahlen von* 1 *bis* 10000 *in ihre Factoren*, Weimar, 1855.
Errors exist* in the Factors of the Numbers below.

804	2708	4452	5546	6240	7076	7518	8365	9170
1026	2894	4454	5568	6264	7101	7534	8688	9496
1354	3120	4504	5592	6291	7112	7598	8794	9596
1566	3132	4558	5610	6344	7192	7632	8901	9606
1608	3216	4748	5616	6478	7274	7770	8904	9688
1836	3332	4918	5788	6664	7316	7796	8908	9768
1848	3672	4975	5800	6768	7344	7815	9008	9836
2052	3696	4984	5805	6785	7392	7946	9024	9875
2374	3885	5416	5847	6810	7456	8003	9032	9897
2442	3898	5454	5973	6867	7458	8144	9116	9950
2508	4104	5475	6003	7034	7505	8208	9160	9968

6. Goldberg's Table,—see pages 29—31.

7. W. J. M. Rankine, *Useful Rules and Tables*, London, 1867.
Tab. 2a, page 33, gives the prime factors of *all* numbers up to 256.
Corrigenda in the Factors-column, [Argument N].

page 34; $N = 195 = 3.5.13$; $N = 248 = 8.31$.

---

* It has been thought sufficient here to merely *locate* the Errors, without giving the Corrigenda.

6. B. Goldberg's, *Primzahlen und Factoren-Tafeln von* 1 *bis* 251617, Leipzig, 1862.

(1) *Corrigenda in the Number-column.*

Page	Line	For	Read	Page	Line	For	Read
5	10 up	4367	4267	169	9	148781	148789
5	2 up	5387	4387	172	7	131409	151409
6	17 up	5939	5039	178	5	156691	156697
7	19	5369	5569	187	13	164776	164779
9	2	7973	7073	189	25 up	166872	166873
13	18	10667	10867	191	21 up	169703	168703
15	2 up	12237	13237	192	6 up	166813	169813
26	21 up	21687	22687	194	21	171084	171083
27	14 up	22669	23669	195	28 up	472121	172121
38	25 up	33347	33247	196	15	172754	172751
40	19 up	34093	35093	196	26	172929	172927
43	18	37517	37417	204	9 up	480377	180377
43	16 up	37899	37799	206	9	191511	181511
44	14	38139	38239	207	21 up	182×48	182849
50	11	42507	43507	208	17	188407	183407
54	4	56939	46939	209	17 up	134673	184673
55	2 up	48627	48629	210	20	195213	185213
60	9	53313	52313	210	26 up	175431	185431
70	19	91321	61321	210	23 up	485471	185471
75	16	65599	65699	211	2	155837	185837
76	24 up	69883	66883	211	22 up	168373	186373
80	4 up	70627	70727	211	1 up	86697	186697
81	25 up	71197	71297	212	30	187138	187139
83	23	12889	72889	212	29 up	157157	187157
83	28 up	73919	73019	215	1	189364	189367
87	2 up	76051	76951	218	26 up	792511	192511
91	14	79729	79829	218	9 up	192760	192769
93	7 up	82481	82181	219	7 up	198681	193681
94	29 up	92733	82733	220	22 up	191339	194339
108	27	95079	95077	222	15 up	106213	196213
108	9 up	95329	95429	223	21 up	196017	197017
109	24	93917	95917	223	13 up	191129	197129
109	28 up	96123	96023	224	22 up	147881	197881
118	26 up	204027	104027	225	16	198434	198439
118	2 up	104287	104387	225	20 up	188791	198791
121	1	106181	106183	227	2 up	200834	200831
123	12	408127	108127	229	11	251893	201893
124	28	409241	109241	231	2	253539	203539
126	17	119857	110857	234	28 up	206638	206639
133	17 up	117438	117433	234	4 up	207001	207007
134	14 up	418367	118367	235	28	907463	207463
139	21	112419	122419	236	1	207943	207947
145	10 up	138159	128159	236	10	298073	208073
154	2	125409	135409	237	26 up	209329	209323
154	20	185673	135673	239	12	110767	210767
154	29	125809	135809	240	14	411673	211673
154	2 up	126241	136241	240	20	211781	211771
157	26 up	198541	138541	240	21	211791	211781
157	4 up	188871	138871	241	3	312407	212407
158	6	439001	139001	242	15 up	212914	213913
166	24 up	146437	146537	244	20	215301	215303

Errata in Goldberg's Table, No. 6—(1)—(*continued*).

Page	Line	For	Read	Page	Line	For	Read
245	4 up	246733	216733	265	27 up	294091	234091
247	10	517811	217811	266	21 up	233067	235067
253	20	223278	223273	267	10 up	236112	236113
255	11 up	225470	225479	268	17	336507	236507
256	16	325863	225863	269	24 up	337671	237671
256	24	225597	225997	270	4	238061	238081
258	18	227668	227669	271	5	238971	238981
259	26	258673	228673	273	10	240820	240829
259	3	223329	228329	273	22	241009	241007
259	8	228403	228409	273	20 up	241274	241271
261	2 up	230928	230923	273	7 up	241468	241469
262	25	231311	231317	274	19	241840	241849
262	25 up	221467	231467	275	13 up	245143	243143
262	3 up	321793	231793	277	2	244241	244249
263	29	232250	232259	278	20	245497	245407
264	6	332801	232801	278	26	245407	245497
264	2	332739	232739	279	2	246917	246017
264	11	238877	232877	280	12	347043	247043
264	13	232801	232901	283	26	249917	249919
264	24	333077	233077	284	25 up	250937	250931
265	10	293741	233741				

Errata in Goldberg's Table, No. 6—(*continued*).

(2) *Corrigenda in the Factors-column*, [Argument N].

Number	Factors	Number	Factors	Number	Factors
5951	11.541	53011	7.7573	93547	139.673
8891	17.523	*53021	37.1433	94001	23.61.67
9571	17.563	*53041	29.31.59	94831	11.37.233
9937	19.523	*53071	73.727	94987	43.47^{2}
11429	11.1039	*53293	137.389	95567	227.421
13559	7.13.149	53731	prime	96301	23.53.79
17651	19.929	53761	37.1453	96631	71.1361
18361	7.43.61	55033	11.5003	96883	17.41.139
19907	17.1171	58729	11.19 281	96937	31.53.59
20009	11.17.107	59807	11.5437	99997	$19^{2}.277$
22919	13.41.43	61807	19.3253	101857	7.14551
23047	19.1213	63631	17.19.197	102283	29.3527
23441	11.2131	64199	43.1493	104303	37.2819
24173	23.1051	64277	17.19.199	105473	29.3637
27347	23.29.41	71623	67.1069	108619	7.59.263
28259	7.11.367	74461	19.3919	108809	53.2053
29597	17.1741	76729	277^{2}	109939	17.29 223
29971	17.41.43	77371	$7^{2}.1579$	111523	229.487
32477	47.691	80357	107.751	113627	$37^{2}.83$
37631	$11^{2}.311$	81617	17.4801	114959	13.37.239
48719	11.43.103	84797	19.4463	116093	17.6829
50813	$7^{2}.17.61$	86483	197.439	118559	7.16937
51209	41.1249	89987	$29^{2}.107$	118859	13.41.223
52693	23.29.79	90419	7.12917		

* The factors of these four numbers are printed so pale (in the copies examined as to be liable to be misread.

Errata in Goldberg's Table, No. 6—(2)—(*continued*).

Number	Factors	Number	Factors	Number	Factors
119287	7.17041	172891	23.7517	215171	11.31.631
119669	11^2.23.43	174247	163.1069	215441	17.19.23 29
119843	37.41.79	174643	7.61.409	216581	19.11399
121033	11.11003	176879	73.2423	216737	73.2969
122483	53.2311	179467	197.911	217039	17^2.751
123763	23.5381	179597	11.29.563	217897	193.1129
124763	17.41.179	179711	7.25673	219209	223.983
128527	7^2.43.61	181561	47.3863	219379	431.509
128851	269.479	182117	13.14009	219859	43.5113
133429	29.43.107	182177	prime	220087	7.23.1367
134057	7.11.1741	182399	7.71.367	220439	17.12967
138379	71.1949	182527	349.523	220993	223.991
140519	83.1693	184423	311.593	221029	83.2663
141187	59.2393	184937	173.1069	223459	19^2.619
142769	11.12979	186083	53.3511	224719	11.31.659
143311	7.59.347	186313	211.883	224729	prime
144859	11.13.1013	186517	37.71^2	225877	107.2111
146189	29.71^2	187537	7.73.367	225901	13.17377
146591	17.8623	187829	31.73.83	226249	61.3709
147017	13.43.263	191423	107.1789	226279	41.5519
147389	11.13399	191839	41.4679	227689	7.11.2957
147581	7.29.727	192203	11.101.173	228967	101.2267
148613	353.421	192449	223.863	229471	11.23.907
149593	227.659	193781	7.19.31.47	229537	7.11^2.271
149603	prime	195151	11.113.157	229579	7.32797
152573	271.563	196301	7.29.967	229907	149.1543
154813	23.53.127	196411	59.3329	230261	19.12119
155011	379.409	197041	13.23.659	231601	31^2.241
155489	61.2549	197501	23.31.277	232427	13.19.941
156263	307.509	198053	23.79.109	233927	223.1049
157117	59.2663	198401	7^2.4049	235093	17.13829
157453	19.8287	198547	36 .541	235801	37.6373
157663	11^2.1303	198617	31.43.149	236099	229.1031
158273	163.971	198947	7.97.293	236281	277.853
158503	31.5113	200167	11.31.587	238271	11.21661
158899	13.17.719	203917	7.29131	239603	7.13.2633
160261	43.3727	204853	11^2.1693	241399	283.853
160283	29.5527	204901	17^2.709	242611	19.113^2
160693	13.47.263	207107	71.2917	242791	97.2503
161299	23.7013	207167	223.929	246863	43.5741
162667	47.3461	207413	211.983	247019	19.13001
165997	13.113	207557	7.149.199	247109	29.8521
166249	83.2003	210217	7.59.509	247751	7.35393
166573	11.19.797	211459	103.2053	247979	17.29.503
169907	131.1297	213793	439.487	248029	97.2557
170951	11 15541	213871	7.30553	249241	47.5303
172231	29.5939	215101	17.12653	251587	7.127.283
172339	23.59.127				

Corrigenda in Col. A. Cunningham's Paper, On Haupt-Exponent Tables, Vol. xxxiii., p. 152.

page 152, *para.* (3),

In Col. of Base (6); *Insert* (under *p*, *e*, *n*) .. 739, 369, 2.

Transfer from Col. of Base (3) to Base (6) (under *p*, *e*, *n*) 997, 332.

PSEUDO-PERIODIC FUNCTIONS ANALOGOUS TO THE CIRCULAR FUNCTIONS.

By the *Rev. F. H. Jackson.*

I.

If we define $\sin x$ by means of its expression as an infinite product, we have

$$\sin x = \frac{\pi}{\Gamma\left(1-\frac{x}{\pi}\right)\Gamma\left(\frac{x}{\pi}\right)},$$

$\Gamma(a+1)$ denoting either

$$\underset{\kappa=\infty}{\text{Lt.}} \frac{1.2.3.4.5\ldots\kappa}{(a+1)(a+2)(a+3)\ldots(a+\kappa)}\kappa^a,$$

or

$$e^{-\gamma a}\prod_{s=1}^{s=\infty}\left\{\frac{1}{1+\frac{a}{s}}e^{\frac{a}{s}}\right\}.$$

From $\sin x$, defined as above, we can derive all the properties of the function and develope a complete theory of the circular functions. Similarly, we can define a function $S_p(x)$ by the equation

$$S_p(x) = \frac{\omega}{\Gamma_p\left(\left[1-\frac{x}{\omega}\right]\right)\Gamma_p\left(\left[\frac{x}{\omega}\right]\right)} \quad \ldots\ldots\ldots(1),$$

in which Γ_p denotes a generalization of Euler's Gamma-function

$$\Gamma_p([a+1]) = \underset{\kappa=\infty}{\text{Lt.}} \frac{[1][2][3][4]\ldots[\kappa]}{[a+1][a+2][a+3]\ldots[a+\kappa]}[\kappa]^a p^{\frac{1}{2}a(a+1)}$$

$$\ldots\ldots\ldots\ldots(2),$$

where

$$[a] = \frac{p^a-1}{p-1},$$

another expression for the same function being

$$p^{\frac{1}{2}a(a-1)}e^{-P_\gamma a}\prod_{s=1}^{s=\infty}\left\{\frac{1}{1+\frac{[a]}{[s]}p^{-a}}e^{\frac{a}{[s]}}\right\} \quad \ldots\ldots\ldots(3),$$

P_γ being an extension of Euler's gamma-constant to the system of numbers [1], [2], [3],..., which we may conveniently term basic numbers. Thus

$$P_\gamma = 1 + \frac{1}{[1]} + \frac{1}{[2]} + \frac{1}{[3]} + \ldots + \frac{1}{[m]} - \log_p \frac{p}{p-1}, \quad p \geqq 1$$

in the limit when m is increased indefinitely. When $p > 1$, the series is absolutely convergent, and the logarithm is finite; but in case $p = 1$, the series is divergent and the logarithm infinite, and the limit is γ.

The function Γ_p has a difference-equation

$$\Gamma_p([a+1]) = [a]\,\Gamma_p([a]),$$

so, referring to the definition of $S_p(x)$, we have

$$\frac{S_p(x)}{S_p(x \pm \omega)} = \frac{\Gamma_p\left(\left[1-\frac{x}{\omega}\right]\right)\Gamma_p\left(\left[\frac{x}{\omega}\right]\right)}{\Gamma_p\left(\left[-\frac{x}{\omega}\right]\right)\Gamma_p\left(\left[1+\frac{x}{\omega}\right]\right)}$$

$$= \frac{\left[\frac{x}{\omega}\right]}{\left[-\frac{x}{\omega}\right]} = \frac{p^{\frac{x}{\omega}}-1}{p^{-\frac{x}{\omega}}-1} = -p^{\frac{x}{\omega}};$$

hence $$S_p(x+\omega) = -p^{\frac{x}{\omega}} S_p(x) \quad \ldots\ldots\ldots\ldots\ldots(4),$$

and $$S_p(x+2\omega) = p^{\frac{2x}{\omega}-1} S_p(x),$$

so that $S_p(x)$ is a pseudo-periodic function singly periodic of the third kind.

Defining now a function $C_p(x)$ by the equation

$$C_p(x) = \frac{\omega}{\Gamma_p\left(\left[\frac{1}{2}+\frac{x}{\omega}\right]\right)\Gamma_p\left(\left[\frac{1}{2}-\frac{x}{\omega}\right]\right)} \quad \ldots\ldots\ldots(5),$$

$$C_p(x) = S_p\left(\frac{\omega}{2}-x\right),$$

also, by means of the difference-equation for the function Γ_p we deduce

$$\frac{C_p(x)}{C_p(\omega+x)} = \frac{\left[\frac{1}{2}+\frac{x}{\omega}\right]}{\left[-\frac{1}{2}-\frac{x}{\omega}\right]} = \frac{p^{\frac{1}{2}+\frac{x}{\omega}}-1}{p^{-\frac{1}{2}-\frac{x}{\omega}}-1} = -p^{\frac{1}{2}+\frac{x}{\omega}},$$

hence $$C_p(\omega+x) = -p^{-\frac{1}{2}-\frac{x}{\omega}} C_p(x),$$

but it has already been shown that

$$S_p(\omega+x) = -p^{-\frac{x}{\omega}} S_p(x),$$

whence $$\frac{S_p(\omega+x)}{C_p(\omega+x)} = p^{\frac{1}{2}} \frac{S_p(x)}{C_p(x)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and therefore $\dfrac{S_p(x)}{C_p(x)}$, the analogue of $\tan x$, is a pseudo-periodic function singly periodic of the second kind.

The zeros of $S_p(x)$ are

$$0, \pm\omega, \pm 2\omega, \ldots,$$

while those of $C_p(x)$ are

$$\pm\frac{\omega}{2}, \pm\frac{3\omega}{2}, \pm\frac{5\omega}{2}, \ldots.$$

In the equation

$$S_p(x) = \frac{\omega}{\Gamma_p\left(\left[\frac{x}{\omega}\right]\right)\Gamma_p\left(\left[1-\frac{x}{\omega}\right]\right)},$$

put $$x = \frac{\omega}{2},$$

then, ω being $\{\Gamma_p([\frac{1}{2}])\}^2$, we have

$$S_p\left(\frac{\omega}{2}\right) = 1, \quad C_p\left(\frac{\omega}{2}\right) = 0$$

and $$S_p\left(\frac{\omega}{4}\right) = C_p\left(\frac{\omega}{4}\right) = \frac{\Gamma_p([\frac{1}{2}])\,\Gamma_p([\frac{1}{2}])}{\Gamma_p([\frac{3}{4}])\,\Gamma_p([\frac{1}{4}])},$$ *

analogous to $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

---

* Which can be expanded in a Hernean series, since

$$\frac{1}{p^{\alpha\beta}}\frac{\Gamma_p(\gamma-\alpha-\beta)\,\Gamma_p(\gamma)}{\Gamma_p(\gamma-\alpha)\,\Gamma_p(\gamma-\beta)} = 1 + \frac{[\alpha][\beta]}{[1][\gamma]}p^{\gamma-\alpha-\beta} + \frac{[\alpha][\alpha+1][\beta][\beta+1]}{[1][2][\gamma][\gamma+1]}p^{2(\gamma-\alpha-\beta)}.$$

$\{\Gamma_p([\frac{1}{2}])\}^2$ is the analogue of π for the basic system of numbers $[1]$, $[2]$, $[3]$, ... as contrasted with the natural system 1, 2, 3, ... This analogue of π enters into the analysis for the functions Γ_p, S_p, C_p, $J_{[n]}$, $P_{[n]}$ in the same way that π does in the analysis of the simpler functions.

II.

Consider the product

$$\Gamma_p\left(\left[1+\frac{x}{\omega}\right]\right)\Gamma_p\left(\left[1-\frac{x}{\omega}\right]\right),$$

which may be written

$$\left\{\frac{[1]}{\left[1+\frac{x}{\omega}\right]}\cdot\frac{[1]}{\left[1-\frac{x}{\omega}\right]}\right\}\left\{\frac{[2]}{\left[2+\frac{x}{\omega}\right]}\cdot\frac{[2]}{\left[2-\frac{x}{\omega}\right]}\right\}\cdots$$

$$\left\{\frac{[\kappa]}{\left[\kappa+\frac{x}{\omega}\right]}\cdot\frac{[\kappa]}{\left[\kappa-\frac{x}{\omega}\right]}\right\}p^{\frac{x^2}{\omega^2}}\cdots(7).$$

In this form the infinite product is absolutely convergent.

We may write

$$S_p(x)=\left[\frac{x}{\omega}\right]\frac{\left[1+\frac{x}{\omega}\right]\left[1-\frac{x}{\omega}\right]}{[1]^2}\cdot\frac{\left[2+\frac{x}{\omega}\right]\left[2-\frac{x}{\omega}\right]}{[2]^2}\cdots$$

$$\frac{\left[\kappa+\frac{x}{\omega}\right]\left[\kappa-\frac{x}{\omega}\right]}{[\kappa]^2}p^{\frac{x^2}{\omega^2}}\cdots(8),$$

$$\log S_p(x+h)=\log\left[\frac{x+h}{\omega}\right]+\sum_{s=1}^{s=\infty}\left\{\log\frac{\left[s+\frac{x+h}{\omega}\right]}{[s]}+\log\frac{\left[s-\frac{x+h}{\omega}\right]}{[s]}\right\}$$

$$+\frac{(x+h)^2}{\omega^2}\log p.$$

Now $$\frac{\left[s+\frac{x+h}{\omega}\right]}{\left[s+\frac{x}{\omega}\right]}=\frac{p^{s+\frac{x+h}{\omega}}-1}{p^{s+\frac{x}{\omega}}-1}=1+p^{s+\frac{x}{\omega}}\frac{\left[\frac{h}{\omega}\right]}{\left[s+\frac{x}{\omega}\right]},$$

so that

$$\log S_p(x+h) - \log S_p(x) = \log\left\{1 + p^{\frac{x}{\omega}} \frac{\left[\frac{h}{\omega}\right]}{\left[\frac{x}{\omega}\right]}\right\}$$

$$+ \sum_{s=1}^{s=\infty}\left\{\log\left(1 + p^{s+\frac{x}{\omega}} \frac{\left[\frac{h}{\omega}\right]}{\left[s+\frac{x}{\omega}\right]}\right) - \log\left(1 + p^{s-\frac{x}{\omega}} \frac{\left[\frac{h}{\omega}\right]}{\left[s-\frac{x}{\omega}\right]}\right)\right\}$$

$$+ \frac{2hx+h^2}{\omega^2}\log p.$$

If now we divide both sides by $\left[\frac{h}{\omega}\right]$ and diminish h indefinitely we obtain

$$\text{Lt.}\frac{1}{\left[\frac{h}{\omega}\right]}\log\frac{S_p(x+h)}{S_p(x)}$$

$$= \frac{1}{\left[\frac{x}{\omega}\right]}p^{\frac{x}{\omega}} + \sum_{s=1}^{s=\infty}\left\{\frac{1}{\left[s+\frac{x}{\omega}\right]}p^{s+\frac{x}{\omega}} - \frac{1}{\left[s-\frac{x}{\omega}\right]}p^{s-\frac{x}{\omega}}\right\}$$

$$+ \text{Lt.}\frac{1}{\left[\frac{h}{\omega}\right]}\cdot\frac{2hx+h^2}{\omega^2}\log p.$$

Now

$$\text{Lt.}\frac{p^{\frac{h}{\omega}}-1}{\frac{h}{\omega}} = \log p,$$

when h is diminished indefinitely, so that

$$\text{Lt.}\frac{1}{\left[\frac{h}{\omega}\right]}\cdot\frac{2hx+\omega}{\omega^2}\log p = \frac{2(p-1)x}{\omega},$$

also we write

$$\text{Lt.}\frac{1}{\left[\frac{h}{\omega}\right]}\log\frac{S_p(x+h)}{S_p(x)} = \frac{\omega(p-1)}{\log p}\text{Lt.}\frac{1}{h}\log\frac{S_p(x+h)}{S_p(x)}$$

$$= \frac{\omega(p-1)}{\log p}\cdot\frac{d\log S_p(x)}{dx}.$$

Finally

$$\frac{\omega(p-1)}{\log p}\cdot\frac{d\log S_p(x)}{dx}$$

$$=\frac{1}{\left[\frac{x}{\omega}\right]}p^{\frac{x}{\omega}}+\sum_{s=1}^{s=\infty}\left\{\frac{1}{\left[\frac{x}{\omega}+s\right]}p^{\frac{x}{\omega}+s}+\frac{1}{\left[\frac{x}{\omega}-s\right]}\right\}+\frac{2(p-1)x}{\omega}$$

.....(9).

This is the analogue of the cotangent series

$$\cot x=\frac{1}{x}+\frac{1}{x+\pi}+\frac{1}{x-\pi}+\frac{1}{x+2\pi}+\frac{1}{x-2\pi}+\ldots.$$

If $p=1$, the series (9) reduces term by term to the cotangent series, since $\underset{p=1}{\text{Lt.}}\frac{p-1}{\log p}=1$.

The series may be written

$$\frac{1}{\left[\frac{x}{\omega}\right]}p^{\frac{x}{\omega}}+\left[\frac{2x}{\omega}\right]\left\{\frac{1}{\left[\frac{x}{\omega}+1\right]\left[\frac{x}{\omega}-1\right]}+\frac{1}{\left[\frac{x}{\omega}+2\right]\left[\frac{x}{\omega}-2\right]}+\ldots\right\}$$

$$+\frac{2(p-1)x}{\omega}\quad\ldots\ldots(10).$$

The ratio

$$\frac{u_{s+1}}{u_s}=\frac{\left[\frac{x}{\omega}+s-1\right]\left[\frac{x}{\omega}-s+1\right]}{\left[\frac{x}{\omega}+s\right]\left[\frac{x}{\omega}-s\right]}=\frac{(p^{\frac{x}{\omega}+s-1}-1)(p^{\frac{x}{\omega}-s+1}-1)}{(p^{\frac{x}{\omega}+s}-1)(p^{\frac{x}{\omega}-s}-1)}.$$

If $p>1$ when s in indefinitely increased $\frac{u_{s+1}}{u_s}=\frac{1}{p}$,

$p<1$ „ „ „ p,

$p=1$ „ „ „ 1,

but special tests show the absolute convergence in this case also.

III.

In this article the theorem analogous to

$$\cot z=\frac{1}{z}-\frac{2z}{\pi^2}S_2+\frac{2z^3}{\pi^4}S_4-\ldots,$$

so that

$$\log S_p(x+h) - \log S_p(x) = \log\left\{1 + p^{\frac{x}{\omega}} \frac{\left[\frac{h}{\omega}\right]}{\left[\frac{x}{\omega}\right]}\right\}$$

$$+ \sum_{s=1}^{s=\infty} \left\{ \log\left(1 + p^{s+\frac{x}{\omega}} \frac{\left[\frac{h}{\omega}\right]}{\left[s + \frac{x}{\omega}\right]} \right) - \log\left(1 + p^{s-\frac{x}{\omega}} \frac{\left[\frac{h}{\omega}\right]}{\left[s - \frac{x}{\omega}\right]} \right) \right\}$$

$$+ \frac{2hx + h^2}{\omega^2} \log p.$$

If now we divide both sides by $\left[\frac{h}{\omega}\right]$ and diminish h indefinitely we obtain

$$\text{Lt.} \frac{1}{\left[\frac{h}{\omega}\right]} \log \frac{S_p(x+h)}{S_p(x)}$$

$$= \frac{1}{\left[\frac{x}{\omega}\right]} p^{\frac{x}{\omega}} + \sum_{s=1}^{s=\infty} \left\{ \frac{1}{\left[s + \frac{x}{\omega}\right]} p^{s+\frac{x}{\omega}} - \frac{1}{\left[s - \frac{x}{\omega}\right]} p^{s-\frac{x}{\omega}} \right\}$$

$$+ \text{Lt.} \frac{1}{\left[\frac{h}{\omega}\right]} \cdot \frac{2hx + h^2}{\omega^2} \log p.$$

Now

$$\text{Lt.} \frac{p^{\frac{h}{\omega}} - 1}{\frac{h}{\omega}} = \log p,$$

when h is diminished indefinitely, so that

$$\text{Lt.} \frac{1}{\left[\frac{h}{\omega}\right]} \cdot \frac{2hx + \omega}{\omega^2} \log p = \frac{2(p-1)x}{\omega},$$

also we write

$$\text{Lt.} \frac{1}{\left[\frac{h}{\omega}\right]} \log \frac{S_p(x+h)}{S_p(x)} = \frac{\omega(p-1)}{\log p} \text{Lt.} \frac{1}{h} \log \frac{S_p(x+h)}{S_p(x)}$$

$$= \frac{\omega(p-1)}{\log p} \cdot \frac{d \log S_p(x)}{dx}.$$

Finally

$$\frac{\omega(p-1)}{\log p}\cdot\frac{d\log S_p(x)}{dx}$$

$$=\frac{1}{\left[\frac{x}{\omega}\right]}p^{\frac{x}{\omega}}+\sum_{s=1}^{s=\infty}\left\{\frac{1}{\left[\frac{x}{\omega}+s\right]}p^{\frac{x}{\omega}+s}+\frac{1}{\left[\frac{x}{\omega}-s\right]}\right\}+\frac{2(p-1)x}{\omega}$$

.....(9).

This is the analogue of the cotangent series

$$\cot x=\frac{1}{x}+\frac{1}{x+\pi}+\frac{1}{x-\pi}+\frac{1}{x+2\pi}+\frac{1}{x-2\pi}+\ldots.$$

If $p=1$, the series (9) reduces term by term to the cotangent series, since $\underset{p=1}{\text{Lt.}}\frac{p-1}{\log p}=1$.

The series may be written

$$\frac{1}{\left[\frac{x}{\omega}\right]}p^{\frac{x}{\omega}}+\left[\frac{2x}{\omega}\right]\left\{\frac{1}{\left[\frac{x}{\omega}+1\right]\left[\frac{x}{\omega}-1\right]}+\frac{1}{\left[\frac{x}{\omega}+2\right]\left[\frac{x}{\omega}-2\right]}+\ldots\right\}$$

$$+\frac{2(p-1)x}{\omega}\quad\ldots\ldots(10).$$

The ratio

$$\frac{u_{s+1}}{u_s}=\frac{\left[\frac{x}{\omega}+s-1\right]\left[\frac{x}{\omega}-s+1\right]}{\left[\frac{x}{\omega}+s\right]\left[\frac{x}{\omega}-s\right]}=\frac{(p^{\frac{x}{\omega}+s-1}-1)(p^{\frac{x}{\omega}-s+1}-1)}{(p^{\frac{x}{\omega}+s}-1)(p^{\frac{x}{\omega}-s}-1)}.$$

If $p>1$ when s in indefinitely increased $\frac{u_{s+1}}{u_s}=\frac{1}{p}$,

$p<1$ „ „ „ p,

$p=1$ „ „ „ 1,

but special tests show the absolute convergence in this case also.

III.

In this article the theorem analogous to

$$\cot z=\frac{1}{z}-\frac{2z}{\pi^2}S_2+\frac{2z^3}{\pi^4}S_4-\ldots,$$

in which

$$S_{2m} = 1 + \frac{1}{2^{2m}} + \frac{1}{2^{2m}} + \ldots ad\ inf$$

will be obtained.

Since $\dfrac{1}{\left[s+\frac{x}{\omega}\right]} p^{s+\frac{x}{\omega}}$ may be expressed as $\dfrac{1}{1+p^s \dfrac{\left[\frac{x}{\omega}\right]}{[s]}} \cdot \dfrac{p^{s+\frac{x}{\omega}}}{[s]}$,

similarly

$$\frac{1}{\left[\frac{x}{\omega}-s\right]} = -\frac{1}{1-\dfrac{\left[\frac{x}{\omega}\right]}{[s]}} \cdot \frac{p^s}{[s]},$$

hence if $\left[\frac{x}{\omega}\right] < 1$, expanding each term in a series of powers $\left[\frac{x}{\omega}\right]$ we obtain the expression

$$\frac{\omega(p-1)}{\log p}\,\frac{d\log(S_p(x)}{dx} = \frac{2(p-1)x}{\omega} + \left[\frac{x}{\omega}\right]^{-1} p^{\frac{x}{\omega}}$$

$$+\left[p^{\frac{x}{\omega}}\left\{p + \frac{p^2}{[2]} + \frac{p^3}{[3]} + \ldots\right\} - \left\{p + \frac{p^2}{[2]} + \frac{p^3}{[3]} + \ldots\right\}\right]$$

$$-\left[p^{\frac{x}{\omega}}\left\{p^2 + \frac{p^4}{[2]^2} + \frac{p^6}{[3]^2} + \ldots\right\} + \left\{p + \frac{p^2}{[2]^2} + \frac{p^3}{[3]^2} + \ldots\right\}\right]\left[\frac{x}{\omega}\right]$$

$$+\left[p^{\frac{x}{\omega}}\left\{p^3 + \frac{p^6}{[2]^3} + \frac{p^9}{[3]^3} + \ldots\right\} - \left\{p + \frac{p^2}{[2]^3} + \frac{p^3}{[3]^3} + \ldots\right\}\right]\left[\frac{x}{\omega}\right]^2.$$

.....

The latter expression being

$$\frac{2(p-1)x}{\omega} + \left[\frac{x}{\omega}\right]^{-1} p^{\frac{x}{\omega}} + \sum_{s=1}^{s=\infty} \frac{p^s}{[s]}\left(p^{\frac{x}{\omega}} - 1\right)$$

$$-\left[\frac{x}{\omega}\right] \sum_{s=1}^{s=\infty} \frac{p^s}{[s]^2}\left(p^{\frac{x}{\omega}+s} + 1\right) + \left[\frac{x}{\omega}\right]^2 \sum_{s=1}^{s=\infty} \frac{p^s}{[s]^2}\left(p^{\frac{x}{\omega}+2s} - 1\right) - \ldots$$

.....(11).

When $p=1$ the coefficients of even powers of $\left[\frac{x}{\omega}\right]$ vanish.

Two series of powers of $\frac{1}{[s]}$ appear, namely

$$\sum_{s=1}^{s=\infty} \frac{p^{ns}}{[s]^n},$$

and

$$\sum_{s=1}^{s=\infty} \frac{p^s}{[s]^n},$$

both absolutely convergent, whether p be $>$ or <1.

Similarly, we may show that

$$\log \frac{S_p(x)}{\left[\frac{x}{\omega}\right]} = (1-p^{-\frac{x}{\omega}})\left[\frac{x}{\omega}\right]\sum_1^\infty \frac{p^s}{[s]} - \tfrac{1}{2}(1+p^{-\frac{2x}{\omega}})\left[\frac{x}{\omega}\right]^2 \sum_1^\infty \frac{p^{2s}}{[s]^2}$$

$$+\tfrac{1}{3}(1-p^{-\frac{3x}{\omega}})\left[\frac{x}{\omega}\right]^3 \sum_1^\infty \frac{p^{3s}}{[s]^3} - \ldots \ldots \ldots (12)$$

corresponding to

$$\log \frac{\sin x}{x} = -2\frac{B_1}{1}\frac{x^2}{2!} - 2^3\frac{B_2}{2}\frac{x^4}{4!} - \ldots .$$

It may be remarked that if $S_p(x)$ be defined as

$$\text{Lt.}\, p^{-\frac{x^2}{\omega}}[x]\frac{[\omega+x][\omega-x]}{[\omega]^2}\frac{[2\omega+x][2\omega-x]}{[2\omega]^2} \ldots ad\ inf.$$

many of the expressions in the paper take a neater form; for example

$$S_p(\omega+x) = -p^{-x}S_p(x), \quad C_p(\omega+x) = -p^{-x-\frac{1}{2}\omega}C_p(x)$$

instead of

$$S_p(\omega+x) = -p^{-\frac{x}{\omega}}S_p(x), \quad C_p(\omega+x) = -p^{-\frac{x}{\omega}-\frac{1}{2}}C_p(x).$$

This form for $S_p(x)$ can be derived from that in the paper by changing the base p to p^ω.

ON THE SOLUTION OF LINEAR DIFFERENCE EQUATIONS BY DEFINITE INTEGRALS.

By *H. A. Webb.*

In his "Théorie Analytique des Probabilités," p. 110, Laplace gives a method for solving linear difference equations by definite integrals. He employs this method, as Boole remarks ("Finite Differences," p. 172), with a view not so much to the solution of difficult equations as to the expression of solutions in forms convenient for calculation when functions of large numbers are involved.

Consider the most general linear difference equation of the nth order for one variable, whose coefficients are linear functions of the variable

$$(a_n\kappa + b_n)\, u_{\kappa+n} + (a_{n-1}\kappa + b_{n-1})\, u_{\kappa+n-1} + \ldots + (a_0\kappa + b_0)\, u_\kappa = 0.$$

Assume as a solution

$$u_\kappa = \int t^\kappa F(t)\, dt,$$

$$\int a_n\kappa \,.\, t^{\kappa+n} F(t)\, dt = a_n \,.\, t^{\kappa+n+1} F(t) - \int a_n \,.\, t^\kappa \frac{d}{dt}[F(t)\, t^{n+1}]\, dt,$$

on integrating by parts.

Hence we have

$$(b_n t^n + b_{n-1} t^{n-1} + \ldots + b_0)\, F(t) = \frac{d}{dt}\{F(t)\, (a_n t^{n+1} + a_{n-1} t^n + \ldots + a_0 t)\},$$

with the condition that

$$[F(t)\, t^{\kappa+1}\, (a_n t^n + a_{n-1} t^{n-1} + \ldots + a_0)]$$

has the same value at both limits of integration.

Therefore

$$\frac{F'(t)}{F(t)} = \frac{\{b_n - (n+1)\, a_n\}\, t^n + (b_{n-1} - n a_{n-1})\, t^{n-1} + \ldots + (b_0 - a_0)}{t\,(a_n t^n + a_{n-1} t^{n-1} + \ldots + a_0)},$$

Suppose the roots of the equation

$$a_n t^n + a_{n-1} t^{n-1} + \ldots + a_0 = 0$$

are all different.

Call them $\alpha_1, \alpha_2, \ldots, \alpha_n$.

$$\frac{F'(t)}{F(t)} = \frac{\beta - 1}{t} + \frac{\beta_1 - 1}{t - \alpha_1} + \ldots + \frac{\beta_n - 1}{t - \alpha_n},$$

$\beta, \beta_1, \ldots, \beta_n$ being coefficients depending on $a_0 \ldots a_n$, $b_0 \ldots b_n$.

Therefore

$$F(t) = Ct^{\beta-1}(t-\alpha_1)^{\beta_1-1}(t-\alpha_2)^{\beta_2-1}\dots(t-\alpha_n)^{\beta_n-1},$$

and

$$u_\kappa = C\int t^{\beta+\kappa-1}(t-\alpha_1)^{\beta_1-1}(t-\alpha_2)^{\beta_2-1}\dots(t-\alpha_n)^{\beta_n-1}dt \quad \dots(1),$$

the integral being taken from

$$t = \alpha_\mu \text{ to } t = \alpha_\nu,$$

if β_μ and β_ν are both positive, otherwise round a "Doppelumlauf" enclosing the points α_μ and α_ν, but no others of the points

$$\alpha_1, \alpha_2, \dots, \alpha_n;$$

where μ and ν have any (different) values from 1 to n.

Hence u_κ has n linearly independent values, which are additive, since the given difference equation is linear.

Hence $$u_\kappa = H_n\begin{pmatrix}\alpha_1, & \alpha_2, & \dots, & \alpha_n, & 0\\ \beta_1, & \beta_2, & \dots, & \beta_n, & \beta+\kappa\end{pmatrix},$$

the hypergeometric function of the nth order, as defined by Pochhammer (*Crelle*, Vol. LXXIII., p. 136).

And u_κ is the general solution of the differential equation

$$\phi(t)\frac{d^n y}{dt^n} - [(\beta+\kappa-n)_1\,\phi'(t)+\psi(t)]\frac{d^{n-1}y}{dt^{n-1}}+\dots$$

$$+(-1)^{n-k}[(\beta+\kappa-k-1)_{n-k}\,\phi^{(n-k)}(t)$$

$$+(\beta+\kappa-k-1)_{n-k-1}\,\psi^{(n-k-1)}(t)]\frac{d^k y}{dt^k}+\dots$$

$$+(-1)^n[(\beta+\kappa-1)_n\,\phi^{(n)}(t)+(\beta+\kappa-1)_{n-1}\,\psi^{(n-1)}(t)]\,y = 0,$$

where
$$\phi(t) = \frac{1}{a_n}(a_n t^n + a_{n-1}t^{n-1}+\dots+a_0)$$

$$= (t-\alpha_1)(t-\alpha_2)\dots(t-\alpha_n),$$

$$\frac{\psi(t)}{\phi(t)} = \frac{\beta_1}{t-\alpha_1}+\frac{\beta_2}{t-\alpha_2}+\dots+\frac{\beta_n}{t-\alpha_n},$$

$$(v)_r \equiv (v-1)\dots(v-r+1),$$

and t is to be equated to zero after the solution has been obtained; it being understood that the n arbitrary constants in the solution are to be replaced by n arbitrary functions, each periodic of period unity.

The case in which the roots of the equation

$$a_n t^n + a_{n-1} t^{n-1} + \ldots + a_0 = 0$$

are not all different has been excluded; the same method of solution is applicable, but the integrand in (1) contains exponentials.

Consider $n = 2$.

The difference equation

$$(a_2\kappa + b_2)\, u_{\kappa+2} + (a_1\kappa + b_1)\, u_{\kappa+1} + (a_0\kappa + b_0)\, u_\kappa = 0,$$

which has been considered by Heymann (*Crelle*, Vol. CXXII., p. 166); or, as it may be written if $a_2 \neq 0$,

$$(\kappa + l + n)\, u_{\kappa+2} - \{(\alpha_1 + \alpha_2)\,\kappa + m\}\, u_{\kappa+1} + \alpha_1\alpha_2\,(\kappa + n)\, u_\kappa = 0$$

is satisfied by the general solution of the differential equation

$$(t-\alpha_1)(t-\alpha_2)\frac{d^2y}{dt^2} - \{(2\kappa + l + 2n - 4)\,t - (\alpha_1+\alpha_2)(\kappa - 2) - m\}\frac{dy}{dt} + (\kappa + n - 1)\,(2\kappa + l + 2n - 4)\, y = 0,$$

where $\alpha_1 \neq \alpha_2$,

t is to be equated to zero after the solution has been obtained, and the two arbitrary constants in the solution are to be replaced by arbitrary functions, each periodic of period unity.

The general integral of this equation is

$$C_1 F\left(a,\, b,\, 2 - \kappa + \frac{m - \alpha_1\,(l + 2n - 4)}{\alpha_1 - \alpha_2},\ \frac{\alpha_1}{\alpha_1 - \alpha_2}\right) + C_2 F\left(a,\, b,\, 2 - \kappa + \frac{m - \alpha_2\,(l + 2n - 4)}{\alpha_2 - \alpha_1},\ \frac{\alpha_2}{\alpha_2 - \alpha_1}\right),$$

where $a + b = -(2\kappa + l + 2n - 3)$

and $ab = (\kappa + n - 1)\,(2\kappa + l + 2n - 4)$.

This agrees with Heymann's result, found by another method.

A modification of the method, also due to Laplace, "Théorie Analytique des Probabilités," p. 123, enables us to solve some equations involving two variables.

For instance, it was proved by Weierstrass (*Mathematische Werke*, Vol. II., p. 253) that the elliptic σ-function $\sigma_\lambda(u)$ [$\lambda = 1$, 2, or 3] is expansible in the form

$$\sigma_\lambda(u) = e^{-\frac{1}{2}\epsilon_\lambda u^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{m,n}\,(12e_\lambda)^m\,(2\epsilon_\lambda)^n\,\frac{u^{2m+4n}}{(2m+4n)!},$$

ϵ_λ denoting $3e_\lambda^2 - \frac{1}{4}g_2$,

where

$$C_{m,n} = 16(m+1)\,C_{m+1,n-1} + n\,C_{m-1,n} - (m+2n-1)(2m+4n-3)\,C_{m,n-1},$$

and $$C_{01} = -1,\quad C_{10} = 0,\quad C_{11} = -1.$$

This equation Weierstrass left unsolved.

Let us write it in the form

$$y_{s,s'} = 16(s+1)\,y_{s+1,s'-1} + s'y_{s-1,s'} - (s+2s'-1)(2s+4s'-3)\,y_{s,s'-1} \quad \ldots(2),$$

and assume

$$y_{s,s'} = \int \kappa^s . \kappa'^{s'} . \phi(\kappa)\, d\kappa,$$

κ' being a function of κ to be determined.

$$\int \kappa^s . \kappa'^{s'} . \phi(\kappa)\, d\kappa \left[1 - 16(s+1)\frac{\kappa}{\kappa'} - s'\frac{1}{\kappa} + (s+2s'-1)(2s+4s'-3)\frac{1}{\kappa}\right] = 0.$$

Let us assume that this integral is expressible in the form

$$\int \phi(\kappa)\, d\kappa [P.F(\kappa) + Q.F'(\kappa) + R.F''(\kappa)] \quad \ldots\ldots(3),$$

where P, Q, and R are independent of κ and κ'.

We find it is so if

$$\frac{\kappa}{\kappa'} \cdot \frac{d\kappa'}{d\kappa} = 2,$$

or $$\kappa' = a\kappa^2,$$

therefore $y_{s,s'} = a^{s'} \int \kappa^{s+2s'} . \phi(\kappa)\, d\kappa_s$, and

$$\int \kappa^{s+2s'} . \phi(\kappa)\, d\kappa \left[1 - 16(s+1)\frac{1}{a\kappa} - s'\frac{1}{\kappa} + (s+2s'-1)(2s+4s'-3)\frac{1}{a\kappa^2}\right] = 0.$$

Let us determine a so that the coefficient of $\frac{1}{\kappa}$ in the bracket may be of the form

$$\tfrac{1}{2}s + s' + \text{a quantity independent of } s \text{ and } s',$$

therefore $a = 32.$

From integration by parts

$$\int\left\{\kappa^{s+2s'}\phi(\kappa) + \tfrac{1}{2}\kappa^{s+2s'+1}\frac{d}{d\kappa}\left[\frac{1}{\kappa}\phi(\kappa)\right] + \tfrac{1}{16}\kappa^{s+2s'-\frac{3}{2}}\frac{d}{d\kappa}\left[\kappa^{\frac{3}{2}}\frac{d\phi(\kappa)}{d\kappa}\right]\right\}d\kappa = 0$$

together with a condition at the limits.

Therefore

$$\frac{d^2\phi(\kappa)}{d\kappa^2} + \left(8 + \frac{3}{2\kappa}\right)\frac{d\phi(\kappa)}{d\kappa} + \left(16 - \frac{8}{\kappa}\right)\phi(\kappa) = 0,$$

and

$$\phi(\kappa) = A_1\int_0^\infty e^{-(u+4)\kappa - \frac{14}{u}}u^{-\frac{1}{2}}\,du + A_2\kappa^{-\frac{1}{2}}\int_0^\infty e^{-(u+4)\kappa-\frac{14}{u}}u^{-\frac{3}{2}}\,du.$$

Substitute this value of $\phi(\kappa)$ in the expression

$$y_{s,s'} = 2^{5s'}\int\kappa^{s+2s'}\phi(\kappa)\,d(\kappa);$$

take 0 and ∞ for the limits of integration, and integrate with respect to κ.

We find

$$y_{s,s'} = B_1.2^{5s'}\Gamma(s+2s'+1)\int_0^\infty e^{-\frac{14}{u}}u^{-\frac{1}{2}}(u+4)^{-s-2s'-1}\,du + B_2.2^{5s'}\Gamma(s+2s'+\tfrac{1}{2})\int_0^\infty e^{-\frac{14}{u}}u^{-\frac{3}{2}}(u+4)^{-s-2s'-\frac{1}{2}}\,du.$$

In a paper on "Generalised Hypergeometric Functions" (*Bulletin of American Math. Soc.*, December, 1903), E. T. Whittaker defines the function

$$W_{k,m}(z)$$

as a particular integral of the differential equation

$$\frac{d^2W}{dz^2} + \left(-\tfrac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4}-m^2}{z^2}\right)W = 0.$$

Adopting his notation, we have

$$y_{s,s'} = C_1 . 2^{-4s-3s'} (2s+4s')\,!\; W_{-s-2s'-\frac{1}{4},-\frac{1}{4}}(\tfrac{7}{2}) + C_2 . 2^{-4s-3s'} (2s+4s')\,!\; W_{-s-2s'-\frac{1}{4},\frac{1}{4}}(\tfrac{7}{2}) \quad \ldots\ldots(4).$$

The second term of equation (3) can also be written

$$C_2 . 2^{-3s-s'} (2s+4s')\,!\; D_{-2s-4s'-1}(\sqrt{7}),$$

where $D_n(z)$, the function of the parabolic cylinder, is a particular integral of the equation

$$\frac{d^2y}{dz^2} + (n + \tfrac{1}{2} - \tfrac{1}{4}z^2)\, y = 0$$

[Whittaker, *loc. cit.*, p. 127].

The equation (4) is a solution of the difference equation (2) containing two arbitrary periodic functions C_1 and C_2, each of period unity in each of the variables s and s', and is the most general solution obtainable on the assumption (3).

The equation (4) is, however, not of much value for computation in the absence of tables of the function $W_{k,m}(z)$. Also it is not the most general solution of the equation (1), and may not include the particular integral sought by Weierstrass.

NOTES ON PLANE CURVES.

By *Harold Hilton.*

I PROPOSE in this paper to collect a few interesting properties of plane curves which I have never seen in print and which are, perhaps, not widely known.

1. *The effect of an ordinary multiple point on the class, &c., of a curve.*

If an ordinary multiple point of order k (a k-ple point say) which has superlinear branches of orders $r, s, t, \ldots, (\Sigma r = k)$ is considered as equivalent to $\{\frac{1}{2}k(k-1) - \Sigma(r-1)\}$ nodes and $\Sigma(r-1)$ cusps, the usual expressions for the class, number of normals through a point, number of inflexions, and deficiency in terms of Plücker's numbers n, δ, κ will hold.*

* Not true for compound multiple points. If the form of a curve at a multiple point is that of $a^{q-p}y^p = x^q$ at the origin (p prime to q), the multiple point may be taken equivalent to $\frac{1}{2}(p-1)(q-3)$ nodes, $(p-1)$ cusps, $\frac{1}{2}(q-p-1)(q-3)$ bitangents, and $(q-p-1)$ inflexions.

In fact the first polar of any point with respect to the curve

$$u \equiv x^r u_{k-r} + u_{k+1} + u_{k+2} + \ldots = 0$$

is of the form

$$x^{r-1} v_{k-r} + v_k + v_{k+1} + \ldots = 0;$$

showing that the curve and its first polar meet at

$$2\{\tfrac{1}{2}k(k-1) - \Sigma(r-1)\} + 3\Sigma(r-1)\}$$

points at the origin. The number of tangents which can be drawn from the multiple point itself (not including the tangents at the multiple point) is $\{2k - \Sigma(r-1)\}$ less than the number which can be drawn from a point not lying on the curve. Again, the inflexions are given by the intersections of $u = 0$ with

$$u_{xx} u^2_y - 2u_{xy} u_x u_y + u_{yy} u^2_x = 0,$$

which is of the form

$$x^{2r-2} w_{2k-3r-2} + w_{2k-3} + w_{2k-2} + \ldots = 0,$$

and meets $u = 0$ at

$$6\{\tfrac{1}{2}k(k-1) - \Sigma(r-1)\} + 8\{\Sigma(r-1)\}$$

points at the origin*. Again, if an n-ic (a curve of the n^{th} degree) has a_2 double points, a_3 triple points, &c., we have

$$\Sigma k(k-1)\, a_k \not> (n-1)(n-2).$$

For otherwise the $(n-2)$-ic having a $(k-1)$-ple point at every k-ple point of the n-ic and passing through

$$\tfrac{1}{2}(n-2)(n+1) - \tfrac{1}{2}\Sigma k(k-1)\, a_k$$

other points on the n-ic would cut the n-ic in more than $n(n-2)$ points. This is not the only limit on the values of the a's†; in particular the multiple points must not be equivalent to more than $\frac{3}{2}(n-2) + 3D$ or more than

$$\tfrac{1}{2}\{4n - 5 + \sqrt{(1+4n)}\} + 2D$$

cusps, where D is the deficiency, since the number of inflexions cannot be negative and the class is $\not< \frac{1}{2}\{1 + \sqrt{(1+4n)}\}$.

---

* A separate investigation is necessary if $s = t = \ldots = 0$, so that u_{k-r} is a constant. The result is, however, unaltered.

† For example, a 5-ic cannnot have two 3-ple points, even though $3.2 \times 2 \not> 4.3$.

2. *On the number of conics satisfying various conditions in connexion with a curve for which Plücker's numbers are* $n, m, \delta, \kappa, \tau, \iota$.

(a) The number of conics through four fixed points which touch the curve is $(2n+m)$.

(b) The harmonic envelopes* of any fixed conic j and $(3n+\iota) = (3m+\kappa)$ conics passing through two fixed points on j and osculating the curve are degenerate.

(c) The harmonic envelopes of any fixed conic j and

$$\tfrac{1}{2}[(2n+m)^2 - 4n - 10m - 3\kappa]$$

conics passing through two fixed points on j and having double contact with the curve are degenerate.

For let h be the locus of the centres of circles touching a fixed curve and orthogonal to a fixed circle j. Then h evidently meets the line at infinity in $(2n+m)$ points, has $2n$ tangents passing through the centre of j, and has one inflexion corresponding to each cusp of the curve. Therefore h has $(3n+\iota)$ cusps and

$$\tfrac{1}{2}[(2n+m)^2 - 4n - 10m - 3\kappa]$$

nodes. The case in which j has zero radius may be treated by inversion; the result is the same as in the general case.

(d) The number of conics through two fixed points having four point contact with the curve is $(5n-3m+3\iota)$†.

If we project any curve on a conicoid from any point v on to any plane, all conics on the conicoid project into conics having double contact with a fixed conic; unless v is on the conicoid, when they project into conics through two fixed points. Therefore to every theorem for conics through two fixed points corresponds a theorem for conics having double contact with a fixed conic. We hence derive from (a), (b), (c), (d):—

---

* The harmonic envelope of two conics is the conic touching all straight lines cut harmonically by the two conics.

† If the curve has a multiple point at any one of the fixed points these results must be modified. For example, if two of the fixed points are k-ple points, the numbers are

(a) $(2n+m-4k)$;

(b) $(3n+\iota-6k)$;

(c) $\tfrac{1}{2}[(2n+m-4k)^2-4n-10m-3\kappa+21k]$;

(d) $(5n-3m+3\iota-8k)$.

(*e*) The number of conics through two fixed points having double contact with a fixed conic j and touching the curve is $2(n+m)$. If j degenerates into a pair of lines* the number is $4(n+m)$: for, projecting the two fixed points into the circules, we see that the locus of the centre of a circle touching the curve and one of the two straight lines l meets l in $2(n+m)$ points, and therefore meets the two bisectors of the angles between the lines in $4(n+m)$ points†. The number of circles touching the curve and two fixed circles is $4(4n+m)$. The number of circles touching the curve and cutting two fixed circles at fixed angles is a problem not distinct from this.

(*f*) The number of conics through a fixed point having double contact with a fixed conic j and osculating the curve is $2(3n+\iota) = 2(3m+\kappa)$.‡ If j is a pair of lines,§ the number is unaltered.‖

(*g*) The number of conics through a fixed point having double contact with a fixed conic j and the curve is

$$[2(n+m)^2 - 2n - 10m - 3\kappa].$$

If j is a pair of lines,† the number is

$$\tfrac{1}{2}[(n+2m)^2 - 10n - 4m - 3\iota].$$‡

(*h*) The number of conics having double contact with j and four-point contact with the curve is

$$3(2n - m + \iota) = 3(2m - n + \kappa).$$

If j is a pair of lines†, the number is $(6n - 4m + 3\iota)$.‡

3. *The satellites of any point with respect to a curve.*

The n real and $n(n-1)$ imaginary intersections of the other circular lines through the intersections of a curve with the circular lines through any point A will be called the satellites of A with respect to the curve. The inverses of the

---

* For the case in which j is a pair of points see (*a*), (*b*), (*c*), (*d*).

† Similarly it is proved that the number of conics through three fixed points touching the curve and a fixed straight line is $2(n+m)$.

‡ (*f*) may be extended thus:—Let j and h be any two conics having double contact with each other; then $2(3n+\iota)$ conics s can be drawn osculating the curve, having double contact with j, and such that the tangents to s and h at two of their intersections are conjugate with respect to j. A similar extension holds for (*g*).

§ For the case in which j is a pair of points see (*a*), (*b*), (*c*), (*d*).

‖ Most easily proved by reciprocation.

CONTENTS.

PAGE

Pseudo-periodic functions analogous to the circular functions. By F. H. Jackson (continued) - - - - - - - - - - - - 38

On the solution of linear difference equations by definite integrals. By H. A. Webb - - - - - - - - - - - - - - 40

Notes on plane curves. By Harold Hilton - - - - - - - 45

Papers for insertion should be sent to Dr. Glaisher, Trinity College, Cambridge, or to Messrs. Metcalfe and Co. Limited, Printing Office, Trinity Street, Cambridge.

An index to vols. i—xxv has been published. It is issued gratis to subscribers, but may be obtained separately, price Two Shillings.

No. CCCC.] NEW SERIES. [August, 1904.

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XXXIV.—NO. 4.

MACMILLAN AND CO.
London and Cambridge.
1904.

Price—One Shilling.

METCALFE AND CO. LIMITED, CAMBRIDGE.

satellites are the satellites of the inverse of A with respect to any inverse curve; unless we invert with respect to A, when the inverses of the satellites of A are the singular foci of the inverse curve. If A is a focus of the curve, four of these satellites coincide in a quadruple (4-ple) satellite of A. If we invert with respect to the 4-ple satellite of a focus A, A inverts into the intersection of inflexional tangents at the circules (a "singular inflexional focus"). If we invert with respect to a focus A, the 4-ple satellite of A inverts into the intersection of tangents to superlinear branches of the second order at the circules (a "singular cuspidal focus"). If, however, the 4-ple satellite of a focus A coincides with A, so that the circular lines through A are tangents to a double point at A (*i.e.* A is a "nodal focus"), the inverse of the curve with respect to A touches the line at infinity at each circle.

Satellites, especially multiple satellites, play an important part in the theory of plane curves; the following theorems will illustrate this statement:—

Given in a bicircular 4-ic with a finite cusp the osculating circle at a fixed point A and the satellite B of A; the locus of the cusp is a bicircular 4-ic touching the given circle and having a focus at B and a node at A. Each branch of the locus at A cuts the given circle at an angle of 30°.

All bicircular 4-ics through four fixed points A, B, C, D, having B as a focus whose satellite is A, pass through a fifth fixed point and have four-point contact with the osculating circle at A.

If two points A, B of a bicircular 4-ic are satellites of each other, the points of contact of the two circles through A and B which touch the curve lie on two circles, one touching the 4-ic at A, and the other at B.

If B is the satellite of a focus A of a bicircular 4-ic with a finite node, the straight line bisecting AB at right angles passes through the intersection of two bitangents of the 4-ic,* &c., &c.

4. *The extended evolute of a curve.*

The evolute of a curve as ordinarily defined is the locus of the finite vertex of the common self-conjugate triangle of any osculating circle, and a circle whose centre is finite and radius infinite. We may extend this idea and consider the

* This theorem (which may be readily proved by inverting with respect to the finite node) gives by projection Mr. Jolliffe's result (*Messenger of Mathematics*, No. 390). His result in *Mess. Math.*, No. 388 may be similarly proved.

locus of the vertices of the common self-conjugate triangle of any osculating conic through two fixed points and a fixed conic through these two points.* It suffices to discuss the locus p of the limiting points of any circle of curvature and a fixed circle j, *i.e.* the envelope of all circles orthogonal to the curve and to j. The locus p is anallagmatic (its own inverse) with respect to j.

The degree of p is $2(3n+\iota)$: for, taking any circle s orthogonal to j, we see that s cuts p at the limiting points of j and the $(3n+\iota)$ circles of curvature orthogonal to s. The locus p meets j at the $2(3n+\iota)$ points of contact of the $2(3n+\iota)$ osculating circles which touch j.

The class of p is $2(2n+4m+\kappa)$; for, by inversion, it is equal to the number of circles through any two points H and K which touch p. Take H and K inverse with respect to j; then the circles through H and K which touch p are the circles through H, K, and the $2(3m+\kappa)$ finite intersections of p and j, and the $(2n+m)$ circles through H and K normal to the curve each of which touches p twice.

The number of inflexions of p = the number of circles through any point which osculate p = the number of circles osculating p and orthogonal to $j = 2(11n+3\iota)$. For three consecutive circles orthogonal to j osculate p at each of the $2(3n+\iota)$ intersections of p and j, and every circular line through an intersection of the given curve and j is an inflexional tangent to p. Also no other circle orthogonal to j can osculate p; for otherwise two consecutive circles orthogonal to the curve and to j would coincide, which is obviously impossible.

We have now found sufficient to determine all Plücker's numbers for the extended evolute.† In some cases these results must be modified. For example, if the given curve has a k-ple point at each circule, the degree, class, and number of inflexions of p are

$$2(2n+\iota-6k),\quad 2(5n+m+\iota-10k),\quad \text{and } 2(11n+3\iota-20k).$$

If the given curve is anallagmatic with respect to j, the numbers are ι, $(m+\iota)$, and 3ι.

5. *Properties of pedal curves.*

Since the r-th negative pedal of a curve is the inverse of the r-th positive pedal of the inverse curve, the properties

* Part of the locus is, of course, the straight line joining the fixed points. We shall exclude this in future.

† For example, the number of cusps is $2(5n-3m+3\iota)$ = twice the number of circles having four-point contact with the given curve, as is otherwise evident.

of negative pedals can be readily obtained when we know the properties of positive pedals: we shall therefore confine our attention to the latter. Properties of positive pedals are most easily obtained by considering the first positive pedal of a curve as the inverse of the reciprocal polar of the curve. Thus to each inflexion of a curve corresponds a cusp of the first pedal, to each cusp corresponds an osculating circle through the pole, &c. The first pedal has, in general, an m-ple point at the pole O (each branch being orthogonal to a tangent from O to the given curve): but if O is a focus of the curve, it is an $(m-2)$-ple point of the pedal. If O is a singular focus, a singular cuspidal focus, a singular inflexional focus, or a nodal focus of the curve it is the same for the pedal. If O is the 4-ple satellite of a focus of the curve, the pedal has inflexions at each circule.

If the curve has a superlinear branch of order k at O or the circules, the first pedal has a superlinear branch of order $(k+1)$, &c., &c. In all cases the deficiency of all the pedals is the same as that of the given curve.

Let n_r, m_r, κ'_r, ι_r, M_r, I_r, f_r denote respectively the degree, class, number of cusps not at O or the circules, number of inflexions, number of tangent circles through any two given points, number of osculating circles orthogonal to a given circle, number of ordinary foci of the r-th pedal of a curve of degree n and class m with ι inflexions.

Then we may readily prove by induction that (if $r>1$)

$$n_r = 2(r-1)n + 2rm, \quad m_r = rn + (r+1)m,$$

$$\kappa'_r = \iota_r = 3n + \iota, \quad M_r = (3r-4)n + (3r-1)m,$$

$$I_r = (2r+1)n + 2rm + \iota, \quad f_r = 0,$$

and that the r-th pedal has at O and at the circules m superlinear branches of order r, and n superlinear branches of order $(r-1)$. If $r=1$, we have $n_1 = 2m$, $m_1 = n + 2m$, $\kappa'_1 = \iota$, $\iota_1 = 3n + \iota$, $M_1 = n + 2m$, $I_1 = 3n + \iota$, $f_1 = n^2$; the first pedal has an ordinary multiple point of order m at O and at each circule.

For example, assuming these statements true for all pedals up to the $(r-1)$-th, we have n_r = twice the degree of the reciprocal polar of the $(r-1)$-th pedal $= 2m_{r-1} = 2(r-1)n + 2rm$.

Again,

$$\kappa'_r = \iota_{r-1} = 3m_{r-1} - 3n_{r-1} + \{\kappa'_{r-1} + 3(r-3)n + 3(r-2)m\}^* = \kappa'_{r-1}.$$

If the given curve has an $\frac{n}{2}$-ple point at each circule, we can show that (if $r > 1$)

$$n_r = 2rm, \quad m_r = (r+1)\,m, \quad \kappa'_r = \iota_r = n + \iota, \quad M_r = (3r-1)\,m,$$
$$I_r = n + 2rm + \iota, \quad f_r = 0,$$

and that the r-th pedal has at $(O\ m)$ superlinear branches of order r, and at each circule $\frac{n}{2}$ superlinear branches of order $(r+1)$, $(m-n)$ of order r, and $\frac{n}{2}$ of order $(r-1)$. If $r = 1$, we have

$$n_1 = m_1 = 2m, \quad \kappa'_1 = \iota, \ \iota_1 = n + \iota, \quad M_1 = n + 2m,$$
$$I_1 = 3n + \iota, \quad f_1 = \tfrac{1}{4}n^2;$$

the first pedal has an ordinary multiple point of order m at O, and at each circule n superlinear branches of order 2 and $(m-n)$ linear branches.

University College,
Bangor.

ON THE HOMOGENEOUS LINEAR DIFFERENCE EQUATION OF THE SECOND ORDER WITH LINEAR COEFFICIENTS.

By *E. W. Barnes.*

§ 1. It is proposed in this paper to examine completely the functional nature of the solution of the linear difference equation

$$(a_2x + b_2)\,f(x+2) + (a_1x + b_1)\,f(x+1) + (a_0x + b_0)\,f(x) = 0 \qquad \ldots\ldots\ldots(O),$$

where the constants a and b have any real or complex values, and where x is the complex variable.

The method of solution has several times been indicated, and may be applied to equations of any order. It was

---

* The superlinear branches of the $(r-1)$-th pedal at O and the circules are equivalent to $3\,(r-3)\,n + 3\,(r-2)\,m$ cusps, cf. § 1.

originally employed by Laplace,* and has been used by Thomae,† Poincaré‡, and Mellin.§ I am induced to publish this paper without attempting to combine it with other investigations in the theory of linear difference equations as Mr. Webb‖ has simultaneously employed the same method to obtain results of a similar though not perhaps of so detailed a nature.

A discussion of the equation with some references has been given by Boole,¶ and a solution has been obtained in a different manner by Heine,** and more recently by Heymann.††

It will be shown that in the general case the solution gives rise to the complete set of hypergeometric functions originally discovered in the theory of linear differential equations by Kummer,‡‡ and to the development of which so much is due to the investigations of Riemann.§§

It is proved that the solution may in all cases be written

$$f(x) = \varpi_1(x) f_1(x) + \varpi_2(x) f_2(x),$$

where $\varpi_1(x)$ and $\varpi_2(x)$ are arbitrary simply periodic functions of x of period unity, and where $f_1(x)$ and $f_2(x)$ are each *uniform* (one-valued) meromorphic functions of x with sequences of poles of similar type at sets of congruent points. By use of the simply periodic multipliers $f_1(x)$ and $f_2(x)$ can, if we choose, always be expressed as integral functions.

As an application the linear difference equations for Legendre's and Bessel's functions are considered.

§ 2. Assume in the first place that in the equation

$$(a_2x+b_2)f(x+2)+(a_1x+b_1)f(x+1)+(a_0x+b_0)f(x)=0 \quad \ldots\ldots\ldots(O)$$

the coefficient a_2 is not zero.

---

* Laplace, *Théorie Analytique des Probabilités*, Oeuvres (1886), pp. 83–86.

† Thomae, *Schlömilch Zeitschrift*, T. 14 (1869), pp. 349–367. A reference to Thomae's work is given in Boole's *Finite Differences*; it seems to be unknown to most writers on the subject, and I had not seen it when the present paper was written. I had hoped that more of the results in the present paper were new.

‡ Poincaré, *American Journal of Mathematics*, Vol. VII., pp. 203 et seq.

§ Mellin, *Acta Societatis Scientiarum Fennicae*, Vol. XX., No. 12.

‖ Webb, *Messenger of Mathematics*, Vol. XXXIV., pp. 40–45.

¶ Boole, *Finite Differences* (1880), Chap. 13.

** Heine, *Kugelfunctionen* (1878), pp. 115–120.

†† Heyman, *Crelle*, Vol. CXXII., pp. 166 et seq.

‡‡ Kummer, *Crelle*, Vol. XV., pp. 39 et seq, and 127 et seq.

§§ Riemann, *Oeuvres Mathematiques* (1898), pp. 61–88; *Mathematische Werke, Nachträge* (1902), pp. 67–93.

Let ω_2 and ω_3 be the roots of the fundamental equation

$$a_2p^2 + a_1p + a_0 = 0,$$

and assume further that $\omega_2 \neq \omega_3$.

Then by use of the substitution

$$\beta_1 + \beta_2 + \beta_3 = \frac{b_2}{a_2},$$

$$\beta_1(\omega_2 + \omega_3) + \beta_2\omega_2 + \beta_3\omega_3 = -\frac{b_1}{a_2},$$

$$\omega_2\omega_3\beta_1 = \frac{b_0}{a_2},$$

which is always possible if $\omega_2 \neq \omega_3$, we may write the equation in the form

$$(x+\beta_1+\beta_2+\beta_3)f(x+2) - \{(x+\beta_1)(\omega_2+\omega_3) + \beta_2\omega_2 + \beta_3\omega_3\}f(x+1) + \omega_2\omega_3(x+\beta_1)f(x) = 0.$$

Put now $x + \beta_1 = \xi$, and let $f(\xi - \beta_1) = \phi(\xi)$.

Then we see that $\phi(\xi)$ will be a solution of

$$(\xi + \beta_2 + \beta_3)f(\xi+2) - \{\xi(\omega_2+\omega_3) + \beta_2\omega_2 + \beta_3\omega_3\}f(\xi+1) + \omega_2\omega_3\xi f(\xi) = 0 \ldots\ldots (P).$$

In this equation let $f(\xi) = \omega_2^{\xi}\phi_2(\xi) = \omega_3^{\xi}\phi_3(\xi)$, and put $k = \frac{\omega_2}{\omega_3}$; then we see that $\phi_2(\xi)$ satisfies the equation

$$(\xi + \beta_2 + \beta_3)f(\xi+2) - [\xi(1+1/k) + \beta_2/k + \beta_3]f(\xi+1) + \xi/k f(\xi) = 0 \ldots\ldots (A),$$

and $\phi_3(\xi)$ the equation

$$(\xi + \beta_2 + \beta_3)f(\xi+2) - [\xi(1+k) + \beta_2 + \beta_3 k]f(\xi+1) + \xi k f(\xi) = 0 \ldots\ldots (B).$$

The equations (P), (A), and (B) may be regarded as normal forms of the general linear difference equation with linear coefficients when the roots of the fundamental equation are not equal.

§ 3. Laplace's method leads us to assume as a solution of (P)

$$f(\xi) = \int t^{\xi} F(t)\, dt.$$

Since

$$\int \xi t^{\xi+n} F(t)\, dt = \int t^{n+1} F(t)\, dt^{\xi}$$

$$= t^{\xi+n+1} F(t) - \int t^{\xi} \frac{d}{dt} [t^{n+1} F(t)]\, dt,$$

we must have

$$\int t^{\xi} \frac{d}{dt} [\{-t^2 + (\omega_2 + \omega_3)\, t^2 - \omega_2 \omega_3 t\}\, F(t)]\, dt$$

$$+ \int t^{\xi} F(t)\, \{(\beta_2 + \beta_3)\, t^2 - (\beta_2 \omega_3 + \beta_3 \omega_2)\, t\}\, dt$$

$$+ t^{\xi+1} F(t)\, [(t - \omega_2)\, (t - \omega_3)] = 0.$$

We must therefore have

$$\frac{F'(t)}{F(t)} = \frac{(\beta_2 + \beta_3 - 3)\, t^2 - (\beta_2 \omega_3 + \beta_3 \omega_2 - 2\omega_2 - 2\omega_3)\, t - \omega_2 \omega_3}{t\, (t - \omega_2)\, (t - \omega_3)}$$

$$= -\frac{1}{t} + \frac{\beta_2 - 1}{t - \omega_2} + \frac{\beta_3 - 1}{t - \omega_3},$$

so that $$F(t) = Ct^{-1} (t - \omega_2)^{\beta_2 - 1} (t - \omega_3)^{\beta_3 - 1};$$

and the integral must be such that $t^{\xi+1} F(t)\, (t - \omega_2)\, (t - \omega_3)$ takes the same value at the beginning and end of the contour of integration.

We thus arrive at a solution

$$f(\xi) = \int t^{\xi-1} (t - \omega_2)^{\beta_2 - 1} (t - \omega_3)^{\beta_3 - 1}\, dt.$$

The branch points of the subject of integration are included in

$$0,\ \omega_2,\ \omega_3,\ \infty.$$

Let a, b denote any two of these four points, and let $\int_{a,b}$ denote an integral taken round what is usually known as a Pochhammer* contour enclosing these points. The contour is continuous and closed: it passes in the positive (counter-clockwise) direction round b, then in the positive direction round a, then in the negative direction round both b and a,

* Jordan, *Cours d'Analyse*, T. 3 (1887). Pochhammer, *Mathematische Annalen*, Bd. 35, 36, and 37 (1889-90).

returning finally to the starting point. It is supposed that the loops of the contour do not enclose other winding points of the subject of integration. When ∞ is one of the two winding points, it is convenient to consider the contour as drawn not on the plane of the complex variable, but on the equivalent Neumann sphere. Then a solution of the normal form (P) of the difference equation will be given by

$$\int_{a,b} t^{\xi-1}(t-\omega_2)^{\beta_2-1}(t-\omega_3)^{\beta_3-1}\,dt.$$

§ 4. Consider in the first place the solution

$$f(\xi)=\int_{0,\omega_2} t^{\xi-1}(t-\omega_2)^{\beta_2-1}(t-\omega_3)^{\beta_3-1}\,dt.$$

Putting $y=t/\omega_2$, we see that a solution of the difference equation (P) is

$$\omega_2^{\xi}\int_{0,1} y^{\xi-1}(1-y)^{\beta_2-1}\left(1-\frac{\omega_2}{\omega_3}y\right)^{\beta_3-1}dy.$$

Now* $\displaystyle\int_{0,1} y^p(1-y)^q\,dy=\frac{-4\pi^2 e^{\pi i(p+q)}}{\Gamma(-p)\,\Gamma(-q)\,\Gamma(p+q+2)}.$

Hence if we put, slightly modifying the notation of Riemann,†

$$P(a,\,b,\,c,\,k)=\int_{0,1} y^a(1-y)^b(1-ky)^c\,dy;$$

we may obtain, if $|k|<1$ and we expand $(1-ky)^c$ by the binomial theorem,

$$P(a,b,c,k)=\frac{2\pi i\,e^{\pi ib}(1-e^{2\pi ia})}{\Gamma(-b)\,\Gamma(-c)}\sum_{r=0}^{\infty}\frac{\Gamma(r-c)\,\Gamma(r+a+1)}{\Gamma(r+1)\,\Gamma(a+b+r+2)}k^r$$

$$=\frac{-4\pi^2 e^{\pi i(a+b)}}{\Gamma(-a)\,\Gamma(-b)\,\Gamma(a+b+2)}F\{a+1,-c,a+b+2,k\},$$

where $F\{\alpha,\beta,\gamma,k\}$ denotes the hypergeometric function

$$1+\frac{\alpha\beta}{1.\gamma}k+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}k^2+\ldots.$$

In this way we arrive at a solution

$f(\xi)=\omega_2^{\xi}P(\xi-1,\ \beta_2-1,\ \beta_3-1,\ k)$ of equation (P),

* *Vide*, for example, Klein, *Hypergeometrische Function* (1894), p. 144.
† *Nachträge*, p. 72.

and therefore at a solution

$$\phi(\xi) = P(\xi - 1,\ \beta_3 - 1,\ \beta_2 - 1,\ k) \text{ of equation } (A).$$

And this latter solution may, when $|k| < 1$, be written

$$\frac{-4\pi^2 e^{\pi\iota(\xi+\beta_2)}}{\Gamma(1-\xi)\,\Gamma(1-\beta_2)\,\Gamma(\xi+\beta_2)} F\{\xi,\ 1-\beta_3,\ \xi+\beta_2,\ k\} \ldots (1),$$

or

$$\frac{\Gamma(\beta_2)\,(1-e^{2\pi\iota\beta_2})\,(1-e^{2\pi\iota\xi})}{\Gamma(1-\beta_3)} \sum_{r=0}^{\infty} \frac{\Gamma(r+1-\beta_3)\,\Gamma(\xi+r)}{\Gamma(r+1)\,\Gamma(\xi+r+\beta_2)} k^r.$$

Since a solution still remains a solution when multiplied by any simply periodic function of ξ of period unity, we may take for the solution of (A), corresponding to the solution

$$\int_{0,\omega_2} t^{\xi-1}(t-\omega_2)^{\beta_2-1}(t-\omega_3)^{\beta_3-1}\,dt \text{ of } P,$$

the series

$$\sum_{r=0}^{\infty} \frac{\Gamma(r+1-\beta_3)\,\Gamma(\xi+r)}{\Gamma(r+1)\,\Gamma(\xi+\beta_2+r)} k^r \ldots\ldots\ldots\ldots (2).$$

This series is a function of ξ, which is expressed as a convergent series of functions of ξ, each of which is uniform and continuous except at the points

$$\xi = -r, \quad r = 0,\ 1,\ \ldots,\ \infty.$$

The series converges uniformly, except in the neighbourhood of these points, and is therefore* a uniform function of ξ, whose sole finite singularities are simple poles at the sets of congruent points

$$\xi = -r,\ r = 0,\ 1,\ \ldots,\ \infty.$$

The alternative solution (1) is a uniform integral function of ξ with no poles in the finite part of the plane.

§ 5. We may readily verify that the series (2) is, under the limitation $|k| < 1$ necessary to secure its convergence, a solution of the difference equation (A).

---

* *Vide*, for instance, Forsyth, *Theory of Functions* (1900), § 83. The nature of the dependence of Riemann's P function upon the indices α, β, γ, p, q, r is discussed by Klein, *Hypergeometrische Function* (1894), p. 188.

For if we substitute for $f(\xi)$ in the left-hand side of the equation, we obtain

$$\frac{1}{k}\left\{\frac{-(\xi+\beta_1)\,\Gamma(1-\beta_3)\,\Gamma(\xi+1)}{\Gamma(\xi+\beta_2+1)}+\frac{\xi\Gamma(1-\beta_3)\,\Gamma(\xi)}{\Gamma(\xi+\beta_2)}\right\}$$

$$+\sum_{r=0}^{\infty}\frac{\Gamma(r+1-\beta_3)\,\Gamma(\xi+r+1)}{\Gamma(r+1)\,\Gamma(\xi+\beta_2+r+1)}k^r\left\{\frac{(\xi+r+1)(\xi+\beta_2+\beta_3)}{\xi+\beta_2+r+1}-(\xi+\beta_3)\right.$$

$$\left.-\frac{(\xi+\beta_3)(\xi+r+1)(r+1-\beta_3)}{(r+1)(\xi+\beta_2+r+1)}+\frac{\xi(r+1-\beta_3)}{r+1}\right\};$$

and we see immediately that this series vanishes term by term.

§ 6. In the next place consider the solution

$$\int_{\omega_3,\,\infty} t^{\xi-1}(t-\omega_1)^{\beta_1-1}(t-\omega_3)^{\beta_3-1}\,dt.$$

By the transformation $y=\omega_3/t$ this becomes, neglecting a factor independent of ξ,

$$\omega_3^{\xi}\int_{0,1} y^{1-\xi-\beta_2-\beta_3}(1-y)^{\beta_3-1}\left(1-\frac{\omega_2}{\omega_3}y\right)^{\beta_2-1}dy,$$

so that a solution of equation (A) is

$$(1/k)^{\xi}\,P(1-\xi-\beta_2-\beta_3,\ \beta_3-1,\ \beta_2-1,\ k).$$

This solution, when $|k|<1$, only differs by a simply-periodic factor from

$$(1/k)^{\xi}\sum_{r=0}^{\infty}\frac{\Gamma(r+1-\beta_3)\,\Gamma(\xi+\beta_2-r-1)}{\Gamma(r+1)\,\Gamma(\xi+\beta_2+\beta_3-r-1)}k^r,$$

or from

$$(1/k)^{\xi}\sum_{r=0}^{\infty}\frac{\Gamma(r+1-\beta_3)\,\Gamma(2-\xi-\beta_2-\beta_3+r)}{\Gamma(r-\xi+2-\beta_3)\,\Gamma(r+1)}k^r.$$

The last series is a one-valued meromorphic function of ξ, whose only singularities are simple poles at the points

$$\xi=-\beta_2-\beta_3+r+2,\quad r=0,\ 1,\ \ldots,\ \infty.$$

The corresponding P function is an integral function of ξ. As before, we may readily verify by substitution that the series satisfies the difference equation.

§7. We have obtained two solutions of the linear difference equation in its normal form (A), each valid when $|k| < 1$. The problem now before us is to classify all the solutions which can be obtained in this manner. We will put

$$a = \xi - 1, \quad b = \beta_2 - 1, \quad c = \beta_1 - 1, \quad d = 1 - \xi - \beta_2 - \beta_3,$$

so that $$a + b + c + d + 2 = 0.$$

The two solutions of equation (A) which we have obtained when $|k| < 1$ correspond to

$$P(a, b, c, k) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and $$(1/k)^\xi P(d, c, b, k) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

We similarly obtain under the same limitation the solutions

$$P(b, a, d, k) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$(1/k)^\xi P(c, d, a, k) \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

When $|k| > 1$ we have the set

$$(1/k)^\xi P(a, c, b, 1/k) \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

$$P(b, d, a, 1/k) \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

$$(1/k)^\xi P(c, a, d, 1/k) \quad \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$P(d, b, c, (1/k) \quad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

When $|1 - k| < 1$, we have

$$(-)^\xi P(a, d, c, 1 - k) \quad \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

$$P(b, c, d, 1 - k) \quad \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

$$(1/k)^\xi P(c, b, a, 1 - k) \quad \ldots\ldots\ldots\ldots\ldots\ldots (11),$$

$$(-1/k)^\xi P(d, a, b, 1 - k) \quad \ldots\ldots\ldots\ldots\ldots\ldots (12).$$

When $|1 - k| > 1$, we have

$$\left(\frac{-1}{1-k}\right)^\xi P\left(a, c, d, \frac{1}{1-k}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (13),$$

$$P\left(b, d, c, \frac{1}{1-k}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (14),$$

$$(1/k)^\xi P\left(c, a, b, \frac{1}{1-k}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (15),$$

$$\left(-\frac{1-k}{k}\right)^\xi P\left(d, b, a, \frac{1}{1-k}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (16).$$

When $\left|\frac{k}{k-1}\right| < 1$, we have

$$\left(\frac{1}{1-k}\right)^{\xi} P\left(a, b, d, \frac{k}{k-1}\right) \quad \text{.....(17)},$$

$$P\left(b, a, c, \frac{k}{k-1}\right) \quad \text{.....(18)},$$

$$(1/k)^{\xi} P\left(c, d, b, \frac{k}{k-1}\right) \quad \text{.....(19)},$$

$$\left(\frac{1-k}{k}\right)^{\xi} P\left(d, c, a, \frac{k}{k-1}\right) \quad \text{.....(20)}.$$

Finally when $\left|\frac{k}{k-1}\right| > 1$, we have

$$(-1/k)^{\xi} P\left(a, d, b, \frac{k-1}{k}\right) \quad \text{.....(21)},$$

$$P\left(b, c, a, \frac{k-1}{k}\right) \quad \text{.....(22)},$$

$$(1/k)^{\xi} P\left(c, b, d, \frac{k-1}{k}\right) \quad \text{.....(23)},$$

$$(-)^{\xi} P\left(d, a, c, \frac{k-1}{k}\right) \quad \text{.....(24)}.$$

§8. The transformations necessary to arrive at these results are given as follows:—

$$y = \frac{t}{\omega_2}, \frac{t}{\omega_3}, \frac{\omega_2}{t}, \frac{\omega_3}{t},$$

$$y = \frac{t}{t-\omega_2}, \frac{t}{t-\omega_3}, \frac{t-\omega_2}{t}, \frac{t-\omega_3}{t},$$

$$y = \frac{t-\omega_2}{-\omega_2}, \frac{t-\omega_3}{-\omega_3}, \frac{-\omega_2}{t-\omega_2}, \frac{-\omega_3}{t-\omega_3},$$

correspond respectively to the solutions

1, 5, 8, 4,

9, 21, 14, 19,

18, 15, 24, 12,

$$y = \frac{t-\omega_2}{\omega_3-\omega_2}, \frac{t-\omega_3}{\omega_2-\omega_3}, \frac{\omega_3-\omega_2}{t-\omega_2}, \frac{\omega_2-\omega_3}{t-\omega_3} \quad \text{to 22, 11, 20, 16,}$$

$$y = \frac{\omega_3(t-\omega_2)}{t(\omega_3-\omega_2)}, \frac{\omega_2(t-\omega_3)}{t(\omega_2-\omega_3)}, \frac{(\omega_3-\omega_2)t}{\omega_3(t-\omega_2)}, \frac{(\omega_2-\omega_3)t}{\omega_2(t-\omega_3)} \quad \text{to 10, 23, 13, 17,}$$

$$y = \frac{\omega_3(t-\omega_2)}{\omega_2(t-\omega_3)}, \frac{\omega_2(t-\omega_3)}{\omega_3(t-\omega_2)}, \frac{t-\omega_2}{t-\omega_3}, \frac{t-\omega_3}{t-\omega_2} \quad \text{to 2, 7, 6, 3.}$$

The solutions can be grouped in equivalent sets of four. Thus to

$\int_{0,\,\omega_2}$	correspond	1, 2, 17, 18,
$\int_{0,\,\omega_3}$	„	5, 7, 13, 15,
$\int_{\omega_2,\,\infty}$	„	6, 8, 14, 16,
$\int_{\omega_3,\,\infty}$	„	3, 4, 19, 20,
$\int_{\omega_2,\,\omega_3}$	„	10, 11, 22, 23,
$\int_{\infty,\,0}$	„	9, 12, 21, 24.

§ 9. In the next place we see that we have

$|k| < 1$ within a circle centre the origin and radius unity,

$|k| > 1$ outside the same circle;

$|1-k| < 1$ within a circle centre the point 1 and radius unity,

$|1-k| > 1$ outside the same circle;

$\left|\frac{k}{1-k}\right| < 1$ to the left of a straight line parallel to the imaginary axis and passing through the point $x = \frac{1}{2}$,

$\left|\frac{k}{1-k}\right| > 1$ to the right of the same line.

Hence some of the solutions are valid for all values of k except those corresponding to the points where the three boundaries of the regions meet, *i.e.* the points $k = e^{\pm \frac{1}{3}\pi i}$.

We have then to consider which, if any, of the solutions we can take when $k = e^{\pm \frac{1}{3}\pi i}$.

Now we have seen that

$$P(a, b, c, k) = \frac{-4\pi^2 e^{\pi\iota(a+b)}}{\Gamma(-a)\,\Gamma(-b)\,\Gamma(a+b+2)}\, F\{a+1, -c, a+b+2, k\}.$$

When $|k| = 1$ the series $F\{a+1, -c, a+b+2, k\}$ is convergent if $\mathfrak{R}(a+1-c-a-b-2)$ is negative, or if $\mathfrak{R}(b+c+1)$ is positive.

We see then that the solutions

1, 4, 5, 8 will be possible if $\mathfrak{R}(\beta_2+\beta_3-1)$ is positive, and

2, 3, 6, 7 „ „ $\mathfrak{R}(\beta_2+\beta_3-1)$ is negative.

In the former case we can take any two of the solutions 1, 4, 5, 8 to be fundamental, in the latter case any two of 2, 3, 6, 7.

§ 10. As an example consider the linear difference equation of Legendre's polynomials, which may be written

$$(x+2)\,f(x+2) - y\,(2x+3)\,f(x+1) + (x+1)\,f(x) = 0.$$

If we compare this with equation (P) we see that

$$\omega_2+\omega_3 = 2y, \quad \xi = x+1, \quad \beta_2+\beta_3 = 1,$$

$$\omega_2\omega_3 = 1, \qquad\qquad \beta_2\omega_2+\beta_3\omega_3 = y.$$

Hence

$$\omega_3 = y - \sqrt{(y^2-1)}, \quad \beta_2 = \beta_3 = \tfrac{1}{2},$$

$$\omega_2 = y + \sqrt{(y^2-1)}.$$

By suitably choosing the sign of $\sqrt{(y^2-1)}$ so that $|k| < 1$ we may take

$$k = \{y - \sqrt{(y^2-1)}\}^2.$$

The solution corresponding to that numbered 4 in the general case is

$$f(x) = \omega_3^{\xi}\, P(1-\xi-\beta_2-\beta_3,\ \beta_3-1,\ \beta_2-1,\ k)$$

$$= \frac{1}{\{y-\sqrt{(y^2-1)}\}^{x+1}}\, P[-1-x, -\tfrac{1}{2}, -\tfrac{1}{2}, \{y-\sqrt{(y^2-1)}\}^2]$$

$$= -\frac{2\pi\iota\,(1+e^{-2\pi\iota x})}{\{y-\sqrt{(y^2-1)}\}^{x+1}}\, \frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)\,\Gamma(\frac{1}{2})} \times F[-x, \tfrac{1}{2}, \tfrac{1}{2}-x, \{y-\sqrt{(y^2-1)}\}^2].$$

Neglecting the simply periodic factor

$$-\frac{2\pi\iota\,(1+e^{-2\pi\iota x})}{y-\sqrt{(y^2-1)}},$$

we see that* this solution is $P_x(y)$.

The solution corresponding to that numbered 1 in the general case is

$$\begin{aligned} f(x) &= \omega_2^{\xi} P(\xi-1,\ \beta_2-1,\ \beta_3-1,\ k) \\ &= \frac{-4\pi^2\omega_2^{\xi} e^{\pi\iota(\xi+\beta_2)}}{\Gamma(1-\xi)\,\Gamma(1-\beta_2)\,\Gamma(\xi+\beta_3)}\, F\{\xi,\ 1-\beta_3,\ \xi+\beta_2,\ k\} \\ &= \frac{-4\pi^2\{y-\sqrt{(y^2-1)}\}^{x+1} e^{\pi\iota(x+\frac{1}{2})}}{\Gamma(\frac{1}{2})\,\Gamma(-x)\,\Gamma(x+\frac{3}{2})} \\ &\qquad \times F[x+1,\ \tfrac{1}{2},\ x+\tfrac{3}{2},\ \{y-\sqrt{(y^2-1)}\}^2] \\ &= 4\sin\pi x e^{\pi\iota(x+\frac{1}{2})}\ Q_x(y), \end{aligned}$$

where

$$\begin{aligned} Q_x(y) = \frac{\Gamma(x+1)\,\Gamma(\frac{1}{2})}{\Gamma(x+\frac{3}{2})}\ &\{y-\sqrt{(y^2-1)}\}^{x+1} \\ &\times F[x+1,\ \tfrac{1}{2},\ x+\tfrac{3}{2},\ \{y-\sqrt{(y^2-1)}\}^2], \end{aligned}$$

in accordance with Heine's definition.

§ 11. It has been stated that in general the fundamental difference equation (O) cannot be reduced to the normal form (A) when $\omega_2=\omega_3$.

In the particular case however when $\omega_2=\omega_3$ in the normal form so that $k=1$, the equation (A) becomes

$$(\xi+\beta_2+\beta_3)\,f(\xi+2)-(2\xi+\beta_2+\beta_3)\,f(\xi+1)+\xi f(\xi)=0.$$

In this case the equation can be very easily solved. For it may be written

$$\frac{f(\xi+2)-f(\xi+1)}{f(\xi+1)-f(\xi)}=\frac{\xi}{\xi+\beta_2+\beta_3},$$

so that

$$f(\xi+1)-f(\xi)=\frac{\Gamma(\xi)}{\Gamma(\xi+\beta_2+\beta_3)}\,\varpi_1(\xi),$$

where $\varpi_1(\xi)$ is a simply periodic function of period unity.

---

* Heine, *Handbuch der Kugelfunctionen* (1878), p. 129.

If now we put

$$f(\xi) = \phi(\xi) \frac{\Gamma(\xi)}{\Gamma(\xi + \beta_2 + \beta_3 - 1)}$$

we obtain

$$\xi\phi(\xi+1) - (\xi + \beta_2 + \beta_3 - 1)\,\phi(\xi) = \varpi_1(\xi),$$

of which the solution is

$$\phi(\xi) = \frac{\varpi_1(\xi)}{1 - \beta_2 - \beta_3} + \frac{\Gamma(\xi + \beta_2 + \beta_3 - 1)}{\Gamma(\xi)}\,\varpi_2(\xi).$$

The complete solution of the equation is therefore

$$f(\xi) = \varpi_1(\xi) \frac{\Gamma(\xi)}{\Gamma(\xi + \beta_2 + \beta_3 - 1)} + \varpi_2(\xi).$$

This accords with the results previously obtained.

For when $k = 1$ and $R(\gamma - \alpha - \beta) > 0$ we know by Gauss' formula that

$$F(\alpha, \beta, \gamma, k) = \frac{\Gamma(\gamma)\,\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\,\Gamma(\gamma - \beta)}.$$

Therefore the former solutions (1) and (4) are both equivalent to

$$\frac{\Gamma(\xi)}{\Gamma(\xi + \beta_2 + \beta_3 - 1)},$$

as they only differ at most by a simply periodic factor from this quotient.

The second principal solution is a constant.

§ 12. Consider now the case when the roots of the fundamental equation (§ 2) $a_2 p^2 + a_1 p + a_0 = 0$ are each equal to ω.

The equation (O) may now, by any obvious substitution, be written

$$(x + \alpha_1 + \alpha_2) f(x+2) - \omega\{2x + 2\alpha_1 + \alpha_3 - \alpha_2\} f(x+1) + \omega^2 (x + \alpha_1) f(x) = 0.$$

If we put $x + \alpha_1 = \xi$ and $f(x) = \omega^x \phi(x + \alpha_1)$ we see that $\phi(\xi)$ satisfies the equation

$$(\xi + \alpha_2) f(\xi+2) - \{2\xi + \alpha_3 - \alpha_2\}\,\phi(\xi+1) + \xi\phi(\xi) = 0 \quad (C),$$

which we take to be the normal form for the present case.

CONTENTS.

PAGE

Notes on plane curves (continued). By HAROLD HILTON - - - - 49

On the homogeneous linear difference equation of the second order with linear coefficients. By E. W. BARNES - - - - - - - - - 52

Papers for insertion should be sent to Dr. Glaisher, Trinity College, Cambridge, or to Messrs. Metcalfe and Co. Limited, Printing Office, Trinity Street, Cambridge.

An index to vols. i—xxv has been published. It is issued gratis to subscribers, but may be obtained separately, price Two Shillings.

No. CCCCI.] NEW SERIES. [September, 1904.

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XXXIV.—NO. 5.

MACMILLAN AND CO.
London and Cambridge.
1904.

Price—One Shilling.

METCALFE AND CO. LIMITED, CAMBRIDGE.

If we assume as a solution of (C)

$$f(\xi) = \int t^{\xi} F(t)\, dt,$$

we obtain

$$\frac{F'(t)}{F(t)} = -\frac{1}{t} + \frac{\alpha_3 - 2}{t-1} + \frac{\alpha_2}{(t-1)^2}.$$

We may therefore express the solution as the integral

$$\int t^{\xi-1} (1-t)^{\alpha_3-2} e^{\frac{\alpha_2}{1-t}}\, dt$$

taken round a suitable contour.

The winding-points of the subject of integration are

$$0,\ 1,\ \infty.$$

We therefore obtain three solutions by taking for the contours of integration Pochhammer circuits enclosing any two of these three points. We may, however, choose different contours.* Divide up the region surrounding the point $t=1$ by a suitable straight line through this point. As t approaches the point $t=1$, the real part of $\frac{\alpha_2}{1-t}$ will tend to $-\infty$ within the area A on one side of this line. The contour may be taken to be any contour enclosing a singularity of the subject of integration, and starting from and ending at $t=1$ within the area A.

§ 13. If we take the solution corresponding to th Pochhammer contour embracing 0, 1, we get

$$f(\xi) = \int_{0,1} t^{\xi-1} (1-t)^{\alpha_3-2} e^{\frac{\alpha_2}{1-t}}\, dt$$

$$= \sum_{r=0}^{\infty} \int_{0,1} \frac{\alpha_2^{\,r}}{\Gamma(r+1)} t^{\xi-} (1-t)^{\alpha_3-r-2}\, dt$$

$$= (1-e^{2\pi i\xi})(1-e^{2\pi i\alpha_3}) \sum_{r=0}^{\infty} \frac{\alpha_2^{\,r}\, \Gamma(\xi)\, \Gamma(\alpha_3 - r - 1)}{\Gamma(r+1)\, \Gamma(\xi + \alpha_3 - r - 1)}.$$

* *Vide*, Nekrassoff, *Mattematische Annalen*, Bd. 38, pp. 509-560; and Klein, *Hypergeometrische Function* (1894), p. 157.

Thus we may take for one solution of (C) the uniform meromorphic function

$$f(\xi, \alpha_2, \alpha_3) = \sum_{r=0}^{\infty} \frac{\alpha_2^r}{\Gamma(r+1)} \frac{\Gamma(\xi)\,\Gamma(\alpha_3 - r - 1)}{\Gamma(\xi + \alpha_3 - r - 1)}.$$

This function has its sole poles at the points

$$\xi = -r, \quad r = 0, 1, \ldots, \infty,$$

When multiplied by $\sin \pi\xi$ it becomes an integral function.

It is very readily established by direct substitution that the series which expresses $f(\xi, \alpha_2, \alpha_3)$ satisfies the difference equation.

A second and independent solution of the equation (C) is given by

$$\int_{1,\,\infty} t^{\xi-1}(1-t)^{\alpha_3-2} e^{\frac{\alpha_2}{1-t}}\, dt.$$

Putting $y = 1/t$, we get

$$\int_{0,\,1} y^{1-\xi-\alpha_3}(1-y)^{\alpha_3-2} e^{\frac{\alpha y}{y-1}}\, dy$$

$$= e^{\alpha_2} \int_{0,\,1} y^{1-\xi-\alpha_3}(1-y)^{\alpha_3-2} e^{\frac{-\alpha_2}{1-y}}\, dy.$$

Thus a second solution is

$$f(-\xi-\alpha_3+2, -\alpha_2, \alpha_3) = \sum_{r=0}^{\infty} (-)^r \alpha_2^r \frac{\Gamma(-\xi-\alpha_3+2)\,\Gamma(\alpha_3-r-1)}{\Gamma(r+1)\,\Gamma(-\xi-r+1)}.$$

This again is a uniform meromorphic function of ξ which can be transformed, if we please, into an integral solution of the difference equation. Other solutions can evidently be obtained by adopting some of the other contours suggested.

The complete solution of the difference equation is now

$$\varpi_1(\xi) f(\xi, \alpha_2, \alpha_3) + \varpi_2(\xi) f(-\xi-\alpha_3+2, -\alpha_2, \alpha_3),$$

where $\varpi_1(\xi)$ and $\varpi_2(\xi)$ are simply-periodic functions of period unity.

When $\alpha_2 = 0$ the solutions become the same to a simply periodic factor. But in this case, as we see at once, one solution is a constant and the other is

$$\frac{\Gamma(\xi)}{\Gamma(\xi + \alpha_3 - 1)}.$$

§ 14. We have still to consider the cases which arise when $a_2 = 0$.

The equation (O) now takes the form

$$b_2 f(x+2) + (a_1 x + b_1) f(x+1) + (a_0 x + b_0) f(x) = 0,$$

and one root of the fundamental equation is infinite.

Assuming that neither a_1 nor a_0 is zero, we can employ the substitutions

$$\frac{b_2}{a_1} = -\frac{\beta_1}{\omega}, \quad \frac{b_1}{a_1} = \beta + \beta_1 + \beta_2,$$

$$\frac{a_0}{a_1} = -\omega, \quad \frac{b_0}{a_1} = -\beta\omega,$$

and write the equation in the form

$$-\frac{\beta_1}{\omega} f(x+2) + (x+\beta+\beta_1+\beta_2) f(x+1) - \omega(x+\beta) f(x) = 0.$$

If now we put $\xi = x + \beta$ and $f(x) = \omega^\xi \phi(\xi)$ we see that $\phi(\xi)$ satisfies the normal equation

$$\beta_1 f(\xi+2) - (\xi + \beta_1 + \beta_2) f(\xi+1) + \xi f(\xi) = 0, \ \ldots \ (D).$$

A solution of this equation is

$$f(\xi) = \int t^{\xi-1} (1-t)^{\beta_2 - 1} e^{-\beta_1 t} \, dt$$

taken round a suitable contour.

One such contour is a Pochhammer contour enclosing the points 0, 1. This contour gives rise to the solution

$$\sum_{r=0}^{\infty} \frac{(-)^r \beta_1^r \Gamma(\xi + r)}{\Gamma(r+1)\,\Gamma(\xi + r + \beta_2)}.$$

This series is a uniform meromorphic function of ξ which may at once be transformed into a solution which is an integral function of ξ.

A second contour we may take to be a contour embracing the origin, and proceeding from and returning to infinity on opposite sides of, and along, the line drawn from the origin to the point $1/\beta_1$ and produced to infinity. It is the usual type of contour by which the Gamma-function is defined.

We take for the second solution

$$\int t^{\xi-1} (1-t)^{\beta_2 - 1} e^{-\beta_1 t} \, dt,$$

and we choose the bulb of the contour, so that $|t|>1$ for all points on the contour.

Now $$\int(-z)^{s-1}e^{-\beta_1 z}\,dz=\frac{-2\pi\iota}{\Gamma(1-s,\beta_1^{\cdot}},$$

the integral being taken along the contour.

Using the binomial theorem we thus obtain the solution

$$\int t^{\xi+\beta_2-2}e^{-\beta_1 t}\sum_{r=0}^{\infty}\frac{\Gamma(r+1-\beta_2)}{\Gamma(1-\beta_2)\,\Gamma(r+1)\,t^r}$$

$$=-2\pi\iota\sum_{r=0}^{\infty}\frac{(-)^{\xi+\beta_2-r-2}\,\Gamma(r+1-\beta_2)}{\Gamma(r+1)\,\Gamma(1-\beta_2)\,\Gamma(r+2-\xi-\beta_2)}\,\frac{1}{\beta_1^{\xi+\beta_2-r-1}},$$

Thus a second solution is

$$\frac{(-)^{\xi}}{\beta_1^{\xi}}\sum_{r=0}^{\infty}\frac{(-)^r\,\Gamma(r+1-\beta_2)}{\Gamma(r+1)\,\Gamma(r+2-\xi-\beta_2)}\,\beta_1^{r}.$$

The complete solution of the equation (D) is therefore

$$\varpi_1(\xi)\sum_{r=0}^{\infty}\frac{(-)^r\beta_1^{r}\,\Gamma(\xi+r)}{\Gamma(r+1)\,\Gamma(\xi+r+\beta_2)}$$

$$+\frac{\varpi_2(\xi)}{(-\beta_1)^{\xi}}\sum_{r=0}^{\infty}\frac{\Gamma(r+1-\beta_2)}{\Gamma(r+1)\,\Gamma(r+2-\xi-\beta_2)}(-\beta_1)^{r}.$$

Each principal solution is, *quâ* function of ξ, one-valued and analytic over the whole ξ plane. The first has poles at the points $\xi=-r$, $r=0,\ 1,\ \ldots,\ \infty$; the second is an integral function.

§ 15. The previous solution breaks down when either $a_1=0$ or $a_2=0$. When both vanish, the equation becomes one with constant coefficients, whose solution is well known.

Consider first the case when $a_1=0$, so that the equation (O) is

$$b_2 f(x+2)+b_1 f(x+1)+(a_0x+b_0)f(x)=0.$$

This equation may evidently be written in the normal form

$$2\omega\phi(\xi+2)-\phi(\xi+1)+\xi\phi(\xi)=0\ldots\ldots\ldots(E).$$

A solution is

$$\int t^{\xi-1}e^{\omega t^2-t}\,dt,$$

taken along a suitable contour.

If we put $t = y^{\frac{1}{2}}$, we obtain as a solution

$$\int e^{\omega y - y^{\frac{1}{2}}} y^{\frac{1}{2}\xi - 1}\, dt,$$

and we may take the contour to be the usual contour for the Gamma-function embracing the line drawn from the origin to $-1/\omega$ and then produced to infinity. Along the contour we may expand $e^{-y^{\frac{1}{2}}}$ and write the solution

$$\int e^{\omega y} \sum_{r=0}^{\infty} \frac{(-)^r}{\Gamma(r+1)} y^{\frac{1}{2}(\xi+r)-1}\, dy,$$

or

$$\sum_{r=0}^{\infty} \frac{(-)^r}{\Gamma(r+1)} \frac{1}{\omega^{\frac{1}{2}(\xi+r)}\, \Gamma\{1 - \frac{1}{2}(\xi+r)\}}.$$

If we put $t = -y^{\frac{1}{2}}$, we obtain as a solution

$$(-)^{\xi} \int e^{\omega y + y^{\frac{1}{2}}} y^{\frac{1}{2}\xi - 1}\, dy = (-)^{\xi} \int e^{\omega y} \sum_{r=0}^{\infty} \frac{y^{\frac{1}{2}(\xi+r)-1}}{\Gamma(r+1)}\, dy.$$

Thus a second solution is

$$(-)^{\xi} \sum_{r=0}^{\infty} \frac{1}{\Gamma(r+1)\, \omega^{\frac{1}{2}(\xi+r)}\, \Gamma\{1 - \frac{1}{2}(\xi+r)\}}.$$

The complete solution of the difference equation (E) is therefore

$$\sum_{r=0}^{\infty} \frac{\omega^{-\frac{1}{2}(\xi+r)}}{\Gamma(r+1)\, \Gamma\{1 - \frac{1}{2}(\xi+r)\}} \{(-)^{\xi} \varpi_1(\xi) + (-)^r \varpi_2(\xi)\},$$

where $\varpi_1(\xi)$ and $\varpi_2(\xi)$ are simply periodic functions of period unity.

Each fundamental solution is an integral function of ξ. The solution may readily be verified by direct substitution. Note that the complete solution remains of the same form when we pass from any one to any other of the values of $\omega^{-\frac{1}{2}(\xi+r)}$.

§ 16. We have finally to consider the case when $a_0 = 0$, and the equation (O) becomes

$$b_2 f(x+2) + (a_1 x + b_1) f(x+1) + b_0 f(x) = 0.$$

By an obvious transformation we may write the equation in the form

$$f(\xi+2) - \frac{2(\xi+1)}{y} f(\xi+1) + f(\xi) = 0 \ldots\ldots (F),$$

which is the difference equation for $J_{\xi}(y)$.

The solution, obtained in the usual way, is

$$f(\xi) = \int e^{-\frac{1}{2}y\left(t-\frac{1}{t}\right)} t^{\xi-1}\, dt,$$

taken along a suitable contour.

We take in the first place the contour to be the Gamma-function contour embracing the line from the origin to $1/y$ produced to infinity.

Expanding under the subject of integration by the exponential theorem, one solution is

$$\begin{aligned}
&\int e^{-\frac{1}{2}yt} \sum_{r=0}^{\infty} \frac{y^r}{2^r \Gamma(r+1)}\, t^{\xi-r-1}\, dt \\
&\quad = -2\pi\iota \sum_{r=0}^{\infty} \frac{(-y)^r}{2^r \Gamma(r+1)\,\Gamma(1+r-\xi)} \frac{(-)^{\xi-1}}{(\frac{1}{2}y)^{\xi-r}} \\
&\quad = 2\pi\iota\,(-)^{\xi} \sum_{r=0}^{\infty} \frac{(-)^r (\frac{1}{2}y)^{-\xi+2r}}{\Gamma(r+1)\,\Gamma(1+r-\xi)} \\
&\quad = 2\pi\iota\,(-)^{\xi} J_{-\xi}(y).*
\end{aligned}$$

Thus one solution may be taken to be $(-)^{\xi} J_{-\xi}(y)$.

To obtain a second solution we may assume an integral of the form

$$\int t^{-\xi} F(t)\, dt.$$

We find on substitution

$$\frac{F'(t)}{F(t)} = \frac{y}{2} - \frac{1}{t} + \frac{y}{2t^2},$$

and hence that a solution is

$$\int e^{\frac{1}{2}y\left(t-\frac{1}{t}\right)} t^{-\xi-1}\, dt,$$

taken along a suitable contour. This contour we may take to be a Gamma-function contour embracing the line from the origin to $-1/y$ produced to infinity. This second solution could obviously be deduced from the former by changing t into $1/t$.

The second solution may be written

$$\begin{aligned}
&\int \sum_{r=0}^{\infty} e^{\frac{1}{2}yt} \frac{(-y)^r}{2^r \Gamma(r+1)}\, t^{-\xi-r-1}\, dt \\
&\quad = 2\pi\iota \sum_{r=0}^{\infty} \frac{(-y)^r}{2^r \Gamma(r+1)} (\tfrac{1}{2}y)^{\xi+r} \frac{1}{\Gamma(1+r+\xi)} \\
&\quad = 2\pi\iota J_{\xi}(y).
\end{aligned}$$

* Nielsen, *Cylinderfunctionen* (1904), p. 5.

Thus for general values of ξ the complete solution of the difference equation (F) is

$$f(\xi) = \varpi_1(\xi)\, J_\xi(y) + \varpi_2(\xi)\, (-)^\xi J_{-\xi}(y).$$

When ξ is an integer n, as is well known, the two solutions are no longer independent. In this case, however, an independent solution is

$$Y_\xi(y) = \operatorname*{Lt}_{\xi=n} \frac{\cos \pi\xi J_\xi(y) - J_{-\xi}(y)}{\sin \pi\xi}$$

$$= \frac{2}{\pi} J_\xi(y) \log \tfrac{1}{2}y - \sum_{s=0}^{\infty} \frac{(-)^s (\frac{1}{2}y)^{\xi+2s}}{\pi \Gamma(s+1)\,\Gamma(\xi+s+1)} \{\psi(s+1)+\psi(s+\xi+1)\}$$

$$- \frac{1}{\pi} \sum_{s=0}^{\xi-1} \frac{\Gamma(\xi-s)}{\Gamma(s+1)} \left(\frac{2}{y}\right)^{\xi-2s}.$$

The complete solution of the difference equation may always be written*

$$f(\xi) = \varpi_1(\xi)\, J_\xi(y) + \varpi_2(\xi)\; Y_\xi(y).$$

Each of the fundamental solutions is a one-valued meromorphic function of ξ.

It is to be noticed that $J_\xi(y)$ and $(-)^\xi J_{-\xi}$ are two different functions of ξ which take the same values at the points $\xi = \pm n$, $(n = 0, 1, 2, \ldots, \infty)$. Regarded, however, as functions of y, they are the same function when the parameter ξ has a positive or negative integral value. Thus we may say that the general functional solution of the difference equation (F) is

$$\varpi_1(\xi)\, J_\xi(y) + \varpi_2(\xi)\, (-)^\xi J_{-\xi}(y);$$

that, when ξ is not an integer, the solution of Bessel's equation is

$$aJ_\xi(y) + bJ_{-\xi}(y);$$

and that when ξ is an integer the general solution of Bessel's equation must be written

$$aJ_\xi(y) + bY_\xi(y).$$

---

* Nielsen (*loc. cit.*), p. 26.

excluded for the reasons there given: this Table shows therefore only those numbers from $(10^7 - 1050)$ to $(10^7 + 1020)$ of the general forms

$N = 10^7 - R = 10^7 - (30m + r)$, where $r = 29, 27, 23, 21, 17, 11, 9, 3$;

$N = 10^7 + R' = 10^7 + (30m + r')$, where $r' = 1, 3, 7, 9, 13, 19, 21, 27$;

all other values of r, r' (< 30) giving N divisible by 2, 3. or 5.

This Table is in *two* parts, one showing $(10^7 - R)$, the other showing $(10^7 + R')$. The arrangement and detail are quite similar to that of Tab. I. and V., and will be understood from the explanation in Art. 6*a*, 15*a*.

16*b*. *High Primes.* Blanks indicate—as in Tab. V.—the *absence* of factors $< \sqrt{N}$, and therefore show *High Primes* (near 10 million). A separate List of these* is given in the short Table below.

List of 118 *High Primes from* $(10^7 - 1050)$ *to* $(10^7 + 1020)$.

9,998,	9,999,...					10,000,...						10,001,
969	...	217	401	653	883	019	253	453	667	789	931	....
971	...	221	419	659	889	079	261	457	687	799	943	...
977	047	233	433	667	901	103	271	481	691	813	961	...
...	049	271	463	677	907	121	303	511	721	819	967	...
...	053	277	469	713	929	139	339	537	723	831	987	...
...	071	289	481	739	931	141	349	583	733	849	993	...
...	083	299	511	749	937	169	357	591	741	867	...	009
...	161	317	533	761	943	189	363	609	747	871	...	...
...	163	337	593	823	971	223	379	643	759	†873	...	...
...	167	347	601	863	973	229	439	651	763	877	...	...
...	193	397	637	877	991	247	451	657	769	891	...	...

16*c*. Use of *Least Factor-Table* (Tab. VI.). See Art. 6*c*, 15*c*. All the composite numbers (N) shown are hereby brought within the power of the large Factor-Tables.

17. *Example* 6°. $N = (10^8 \mp N)$. The number here chosen as "Central Number" is $N_0 = 10^8 = 100$ million, and the "Range" chosen is the group of 4000 numbers from $(10^8 - 2000)$ to $(10^8 + 2000)$.

17*a*. *Least Factor-Tables* (Tab. VII*a*, *b*, *c*, *d*; VIII*a*, *b*, *c*, *d*). These Tables are very similar to the preceding ones (Art. 15*a*, 16*b*), except that they show *all*‡ numbers (N) within the given "Range" $(10^8 \mp 2000)$, and

* These primes were *announced* at the British Association Meeting of 1903), (see Brit. Assoc. Report of 1903, p. 561), but have *not been printed* in the Report.

† The number 10,000,873 had been previously determined *prime*, see the Abstract of a Paper on *Idoneal Numbers* by Lt.-Col. A. Cunningham and the Rev. T. Cullen, in Brit. Assoc. Report, 1901, p. 552.

‡ Numbers containing 2, 3, 5 cannot be omitted in the present case, because the quotient $N \div \mu$ is beyond the power of the existing Factor Tables, when $\mu = 2, 3, 4, 5, \ldots, 11$.

that the Tables of $(10^8 - R)$ have been drawn up so as to read *additively* ($N = 99{,}998{,}000 + R$, in Tab. VII*a*, *b*; $N = 99{,}999{,}000 + R$, in Tab. VII*c*, *d*), so that each number (N) can be read off at sight. These Tables are drawn up in *eight* parts (on separate pages), each containing 500 numbers (N as below); [$r = 0$ to 9].

Tab.	VII*a*; VII*b*	VII*c*; VII*d*	VIII*a*; VIII*b*	VIII*c* ; VIII*d*
$N =$	$99998000+10m+r$	$99999000+10m+r$	$100000000+10m+r$	$100000000+10m+r$
$10m =$	0to490 ; 500to990	0to490 ; 500to990	0 to 490 ; 500 to990	1000 to 1490 ; 1500 to 1990

The body of the Tables shows for all the composite numbers (N) at least one prime factor, viz. always the *lowest* prime factor > 5, or power of that factor (if repeated): the factors 2, 3, 5 and their powers are so easily recognised that it has not been thought worth while entering them at all (except in one case noted below), the space saved being thus available for other divisors. Also, when the width of the column (5 figures) admits, the residual factor is indicated (if *prime*) by the letter q or Q, viz. by q if < 9 million, and by Q if > 9 million (so that Q denotes a *High Prime*): and in this latter case the small factors 2, 3, 4, 5, 2.3, 8, 9, 2.5, (as well as 7, 11), are printed, thereby indicating the magnitude of Q (the saving of space not being required in this case).

Use of $\sigma = 7$, $\varepsilon = 11$. When $N \neq 2n$, $3n$, or $5n$, and has 7 or 11 as "least factor," followed by a 3-, or 4-figure factor, then contractions are used, σ for 7, and ε for 11, when required to include the *two* factors within the 5-figure width of column.

17*b*. *High Primes.* Blanks in the above Tables indicate the absence of factors $< \sqrt{N}$, and therefore show *High Primes* (> 99 million in Tab. VII*a*, *b*, *c*, *d*; > 100 million in Tab. VIII*a*, *b*, *c*, *d*): also the letter Q indicates *Secondary High Primes* (> 9 million, but < 51 million). Or, briefly, all numbers N in these Tables containing no factor (μ) > 11 yield *High Primes*, $p = N \div \mu$ ($\mu = 1$ to 11). A separate List of these* High Primes is given in the Table below; the following is an Abstract of the total number (n) of these High Primes (p) for each divisor (μ); the last line shows the approximate magnitude of each class in millions (M denoting *one million*).

$\mu =$	1	2	3	4	5	6	7	8	9	10	11	1 to 11
$n =$	209	122	75	57	48	41	29	25	28	22	23	679
p near	$100M$	$50M$	$33\frac{1}{3}M$	$25M$	$20M$	$16\frac{2}{3}M$	$14\frac{2}{7}M$	$12\frac{1}{2}M$	$11\frac{1}{9}M$	$10M$	$9\frac{1}{11}M$	$> 9M$

The total number of High Primes shown in this Table, and in the Abstract is 679; but 22 of them, viz. all of forms $p = \frac{1}{10}N$ appear also in the Table of Art. 16*b*, so that only 657 are *new*.

---

* Some of the primes in this List, viz. those of form $p = N \div \mu$, [$\mu = 1$ to 11, $N = 99998500$ to 10^8] were *announced* at the British Association Meeting of 1903, (see Brit. Assoc. Report of 1903, p. 561), but have *not been printed* in the Report. For *earlier* publication of 91 primes $> 10^8$, and of 11 primes $= \frac{1}{11}(10^8 + R)$, see Art. 17*d*, *e*, below.

List of 679 *High Primes* $p = \frac{1}{\mu} N$; *between* $N = (10^6 \mp 2000)$; [$\mu = 1$ *to* 11.]

$\mu = 1$								$\mu = 2$			
99,998,	99,998,	99,999,	99,999,	100,000,	100,000,	100,001,	100,001,	49,999.	49,999,	50,000,	50,000,
	377	043	563		463	029	611	031	597	017	507
	413	073	587		469	053	623	069	603	021	513
	417	077	589		471	059	651	111	613	047	537
	429	079	611		493	081	653	121	619	059	539
	441	089	617		541	087	687	133	627	063	557
	447	103	623		543	107	689	151	637	101	563
	449	113	643		561	119	719	177	639	131	567
	473	131	677		567	131	761	207	643	141	569
	491	157	703	007	577	147	767	231	667	161	617
	497	167	721	037	609	159	777	253	673	201	627
	513	187	773	039	627	177	791	267	693	207	641
	543	217	787	049	643	183	801	289	699	221	683
	551	247	821	073	651	203	809	291	711	231	711
	557	257	827	081	661	207	813	297	739	233	719
	603	259	839	123	669	219	819	307	751	239	729
021	609	307	847	127	673	227	821	349	753	243	747
051	611	323	931	193	687	303	827	351	757	257	773
071	663	329	941	213	717	329	843	361	759	273	813
099	749	343	959	217	721	333	887	387	777	309	849
111	783	353	971	223	793	347	893	403	783	311	851
147	797	373	989	231	799	357	903	409	801	329	857
191	819	401		237	801	399	927	423	819	341	897
207	851	437		259	837	431	929	427	843	347	921
231	869	439		267	841	449	983	441	847	387	947
251	891	481		279	853	467	987	463	853	389	953
257	909	509		357	891	507		471	877	393	971
291	929	517		379	921	533		489	883	399	981
299	933	539		393	937	537		529	897	429	983
323	953	541		399	939	569		553	903	449	999
357	971	547		421	963	581		561	921	477	
363	999	551		429	969	591		589	991	491	

List of 679 High Primes $p = \frac{1}{\mu} N$, *between* $N = 10^6 \mp 2000$; $[\mu = 1 \text{ to } 11]$. *(Continued).*

$\mu = 3$			$\mu = 4$		$\mu = 5$		$\mu = 6$		$\mu = 7$	$\mu = 8$	$\mu = 9$	$\mu = 10$	$\mu = 11$
33,332,	33,333,	33,333,	24,999,	25,000,	19,999,	20 000,	16,666,	16,666,	14,285,	12,499,	11,110,	9,999,	9,090,
	019	431		009		003	343	877	449	757	889	823	749
	031	‡473		033		023	357	879	459	801	901	863	751
	037	493		069		‡033	361	883	471	813	903	877	791
	061	523	511	109		047	367	891	477	867	907	883	793
	067	‡529	521	111		059	369	907	519	871	921	889	811
	071	551	523	127		063	381	921	521	873	933	901	821
	077	571	529	133		‡069	417	931	533	897	937	907	847
	119	589	563	139		‡077	423	957	543	901	943	929	853
	133	611	599	141		‡081	453	961	561	909	963	931	857
	137	631	607	147		‡093	487	997	563	919	999	937	859
	143	637	617	163		107	493		569	987		943	871
	161	667	619	169		147	501		581	999	11,111,	971	901
683	191	683	661	177		‡153	523		597			973	§919
711	193	749	697	207		159	547		639	12,500,	039	991	947
723	203	791	703	211	631	‡161	549		659		053		‡953
777	217	829	719	231	663	171	577		669	003	069	10,000,	973
807	241	851	731	267	697	213	583		693	029	083		989
839	247	887	757	273	729	221	589		717	069	099	019	
851	253	899	761	277	739	243	597		731	071	101	079	9,091,
857	257	919	803	279	789	269	607		741	077	‡117	103	
863	269	931	811	303	817	287	609		749	087	119	121	009
881	281	941	841	321	831	297	621		837	093	‡131	139	021
899	283	947	847	351	843	303	631		857	†101	‡141	141	051
909	319	967	859	363	873	311	669		879	177	‡161	169	057
927	329	977	889	373	891	327	711		923	197	‡171	189	067
933	331		929	411	897	329	799		927	221	‡173		§079
941	347		959	427	909	339	807		977	227	‡201		
947	359		967	439	927	353	823		983	231	267		
953	373		971	487	963	359	829		989		273		
987	403		973		981	377	841				311		
993	427		983		*999	389	873				327		

* The number 19999999 had been previously determined *prime*, see the present authors' Solutions of Q. 14305 in *Mathematical Questions with their Solutions from the Educational Times*, Vol. LXXIII., 1900, p. 89.

† The number 12500101 had been previously determined *prime*, see Ed. Lucas's Memoir *Sur la série récurrent de Fermat*, Rome, 1878, p. 7.

‡ *See* Art. 17*e*—ii. § *See* Art. 17*e* – i.

17*c*. *Use of Least Factor-Tables* (Tab. VII*a*, *b*, *c*, *d*; VIII*a*, *b*, *c*, *d*). See Art. 6*c*, 15*c*. All the composite numbers (N) shown are hereby brought within the power of the large Factor-Tables.

17*d*. *High Primes* $> 10^8$. [Others' work]. The High Primes just $> 10^8$ have been determined by two other authors, viz.

i. *By Mr. Wm. Davis.* In the *Journ. de Math. pures et appl.*, 2° Sér., t. xi. 1866, there is a short paper,—(apparently an Editorial Note)—styled *Les Nombres premiers de* 100000001 *à* 100001699*, containing a List of *all* the High Primes between 10^8 and ($10^8 + 1700$†) determined by Mr. W. Davis: the number printed is 99, but *eight* of these are really *composite*, viz.

$N = 10^8 + 0013,\ 0391,\ 0657,\ 0723,\ 1221,\ 1353,\ 1549,\ 1647.$

Least factor = 827, 2153, 2711, 1447, 2689, 7247, 2447, 5737.

The remaining 91 agree with the present authors' work.

ii. *By Mr. T. B. Sprague.* The High Primes between 10^8 and ($10^8 + 1000$) have been quite recently determined also by this author. His MS. List‡ of 54 primes within this "Range" agrees with the present author's work.

Thus the 54 High Primes between 10^8 and ($10^8 + 1000$) have now been determined by *four* computers (working independently), and the 37 High Primes between ($10^8 + 1000$) and ($10^8 + 1700$) have been determined by *three* computers (working independently): so that these primes may now be accepted with some§ confidence.

17*e*. *High Primes*, $p = \frac{1}{\mu} \cdot (10^8 + R)$. [Others' Work].

i. *By Mr. Wm. Davis.* There is a short Note by this author, styled "153 *primes of the* 10th *million*," in *Math. Questions with Solutions from the Educational Times*, Vol. VIII., 1868, page 30, consisting *merely of a List* of 153 primes in the 10th million (without any description or explanation): it confirms the present authors' determination of the *eleven* primes of form $p = \frac{1}{11}(10^8 + R)$ between the marks§ in the Table on p. 77.

ii. *By Mr. T. B. Sprague.* Some of the High Primes, $p = (10^8 + R) \div \mu$, where $R < 1000$, and $\mu = 3, 5, 9, 11$, have also been recently determined by this author. The MS. List‡ confirms the 17 High Primes marked ‡ in the Table above, (page 77).

18. *Tests of Work.* Same as in Art. 9, *q.v.*; also add

Ex. 4°, 5°, 6°. The whole of the arithmetical work has been worked out by two computers‖ *independently*; the results were then carefully collated, and all discrepancies were further examined.

---

* Misprinted 100001169 in *Mess. of Math.*, Vol. XXXI., 1902, p. 171 (footnote †).

† Misprinted 1170 in *Mess. of Math.*, Vol. XXXI., 1902, p. 171, Art. 10.

‡ Kindly lent for the purpose of this verification.

§ The eight mistakes now discovered in Mr. Davis's List afford a good instance of the necessity of such laborious calculations being worked by at least two computers (working independently).

‖ viz. by Miss A. Woodward (under Col. Cunningham's superintendence) and Mr. H. J. Woodall. For description of a new (and very simple) mode of effecting the divisions required, see a Paper *On Synthetic Division*, p. 90 of this Journal, by Mr. Woodall.

19. *Sequences of composites.* Add to previous Art. 11 the following *long sequences of composite numbers* (without any prime intervening) :—[here M denotes *one million*.]

Two of 51 in $9M$ to $(9M+1000)$	one of 87 in $(16\frac{2}{3}M \mp 333)$
one of 69 & of 77 in $(10M \mp 1020)$	one of 65 in $(33\frac{1}{3}M \mp 666)$
one of 67 & of 75 in $(12\frac{1}{2}M \mp 250)$	one of 69 in $(50M \mp 1000)$
one of 87 in $(14\frac{1}{7}M \mp 285)$	one of 71, 75, 77, 83, 85 in $(100M \mp 2000)$.

20. *Distribution of primes.* Add to previous Art. 12 the following numbers (n) of primes within the Range of 1000 numbers from $(N_m - 1000)$ to N_m. Here M denotes *one million*].

$N_m = 9M + 1000,\ 10M,\ (25M + 500),\ (33\frac{1}{3}M + 500),\ 50M,\ 100M,$

$n =$ 60 , 53 , 57 , 61 , 62 , 52.

21. *Prime-pairs.* A pair of primes p, q, such that $p \sim q = 2$, is sometimes called a *Prime-pair** : the number of such pairs within a given "Range" is of some interest. The following Table shows the number (n') of such prime-pairs in the "Ranges" stated on either side of the Central Number (N). [Here M denotes *one million*]. The highest of such prime-pairs now known is believed to be 100,001,927, 100,001,929. The last line shows the number (n) of primes within the same Range as n', for comparison with n'.

$N_0 =$	$9M$	$9\frac{1}{11}M$	$10M$	$11\frac{1}{9}M$	$12\frac{1}{2}M$	$14\frac{1}{7}M$	$16\frac{2}{3}M$	$20M$	$25M$	$33\frac{1}{3}M$	$50M$	$100M$
Range	$+1000$	∓ 181	∓ 1000	∓ 222	∓ 250	∓ 285	∓ 333	∓ 400	∓ 500	∓ 666	∓ 1000	∓ 2000
$n' =$	4	3	8	4	2	2	4	2	6	3	12	16
$n =$	60	23	114	28	25	29	41	48	57	75	122	209

This bears out (in a general way) Dr. Glaisher's statement† that "as we ascend higher in the series of numerals, the number of prime-pairs must decrease also"; meaning thereby the number contained in a given interval (when that interval is large). It also bears out (in a general way) another suggestion‡ of the same author that "the numbers of prime-pairs * * * are rather less than one-tenth of the number of primes in the same intervals"; here again the *intervals* must be large.

[Here follow Tables of *Least Factors*, (Tab. V., VI., VII*a*, *b*, *c*, *d*; VIII*a*, *b*, *c*, *d*). For explanation, *see* Arts. 15*a*, *b*, *c*; 16*a*, *b*, *c*; 17*a*, *b*, *c*.]

---

* See *An Enumeration of Prime Pairs* by J. W. L. Glaisher, in *Messenger of Mathematics*, Vol. VIII., 1879, p. 28.

† *op. cit*, p. 28. ‡ *op. cit.*, p. 31.

Least factors of $9.10^6 + (30m + r')$. Table V.

30m	r' = 1	7	11	13	17	19	23	29
0	61q	277q		11.31	13q	7q	239q	43
30	71q	59		13.67	7.149		19	
60	7.23q		643q	2393	47q	11q	2689	7.17
90	10.19	73q	121q	7.461	23	127q	379q	
120	13.29		7.167	1667	31q	37		281q
150	1823	17q	907q		11.19	193q	49.13	29
180	2719	7q	17q		41q	13.23		
210	11		331q		863q	7q	11.79	109q
240	8.27q	487q	13q		7.403	17	2411	
270	40.53	11.13	19		37.43		17	7.11
300	53 q	5.35q	67q	7q		19	31	13q
330		43	7q	121q	761q		29q	47q
360	17.37	61q	1039q	43q			7.23q	73
390	50.73	7.25?	1181		13.53	11q	101q	883q
420	1003		11	13.13	83q	7q	41q	1.5q
450		71		17	43q	29q		1907
480	7.80q		457q	101q	11.37		223q	7.43
510	13q		43	7q	29q	1033	67q	4739
540	11q	19.47	7q	1451	2539	421q	11.23	
570	31		317q	37q	13 q	13q	7q	17.79
600		7.11	61q	23q	127	53q	19	7.6
630	43q		13.47	29	509q	7.203	157q	23.89
660	109	13.17	17.3q	11	7q	61.56		4.16
690	7q	103q	17.29		79q		17q	7.179
720		67q	37q	7.11	89q	113q		1531
750	83q	5 q	401.	157q		17.83		
780					13.53	127q	7.17	
810	1487	7.29		79q	121	47q	347q	
840		17q	109	4.7q	1.39	401	211	2011q
870	11.17		3 q	11	7q	1.433		
900	7 139		2.7q		1001	7.67q	47q	7q
930		17.53	14.1	7.31	127q	7.1	13.27	1.1
960		157q	7q	17.73	1 q	19		
990	89q			1.1.1	17q	2109q	7q	1111q

CONTENTS.

PAGE

On the homogeneous linear difference equation of the second order with linear coefficients (continued). By E. W. BARNES - - - - - - - 65

Determination of successive high primes. By LT.-COL. CUNNINGHAM and H. J. WOODALL - - - - - - - - - - - - - 72

Papers for insertion should be sent to Dr. Glaisher, Trinity College, Cambridge, or to Messrs. Metcalfe and Co. Limited, Printing Office, Trinity Street, Cambridge.

An index to vols. i—xxv has been published. It is issued gratis to subscribers, but may be obtained separately, price Two Shillings.

No. CCCCII.] NEW SERIES. [October, 1904.

Stanford University Library
NOV
30
1904

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XXXIV.—NO. 6.

MACMILLAN AND CO.
London and Cambridge.
1904.

Price—One Shilling.

METCALFE AND CO. LIMITED, CAMBRIDGE.

Least factors of $9.10^6 + (30m + r')$. Table V.

$30m$	r'							
	1	7	11	13	17	19	23	29
.	61*q*	277*q*		11.31	13*q*	7*q*	239*q*	43
30	71*q*	59		13.67	7.149		19	
60	7.23*q*		643*q*	2393	47*q*	11*q*	2689	7.17
90	19.19	73*q*	121*q*	7.461	23	127*q*	379*q*	
120	13.29		7.167	1667	31*q*	37		281*q*
150	1823	17*q*	907*q*		11.19	193*q*	49.13	29
180	271*q*	7*q*	17*q*		41*q*	13.23		
210	11		331*q*		863*q*	7*q*	11.79	109*q*
240	827*q*	487*q*	13*q*		7.463	17	2411	
270	49.83	11.13	19		37.43		17	7.11
300	53 *q*	523*q*	67*q*	7*q*		19	31	13*q*
330		23.	7*q*	121*q*	761*q*		29*q*	47*q*
360	17.37	61*q*	163*q*	43*q*			7.23*q*	73
390	59.79	7.257	1181		13.53	11*q*	101*q*	883*q*
420	1063		11	13.13	83*q*	7*q*	41*q*	103*q*
450		71		17	49*q*	29*q*		1997
480	7.89*q*		467*q*	191*q*	11.17		223*q*	7.19
510	13*q*		23	7*q*	29*q*	1033	97*q*	419*q*
540	11*q*	19.47	7*q*	1451	233*q*	421*q*	11.13	
570	31		317*q*	37*q*	181*q*	13*q*	7*q*	17.71
600		7.11	61*q*	23*q*	127	53*q*	19	11*q*
630	43*q*		13.47	29	509*q*	7.263	157*q*	23.89
660	19*q*	13.17	173*q*	11	7*q*	613*q*		41*q*
690	7*q*	163*q*	17.29		79*q*		67*q*	7.13*q*
720		97*q*	37*q*	7.61	19*q*	11*q*		1531
750	23*q*	31*q*	49.11	137*q*		17.83		
780					13.23	167*q*	7.17	
810	1487	7.29		13*q*	121	47*q*	347*q*	
840		67*q*	19*q*	41*q*	1039	499	59	2063
870	11.17		31*q*	71	7*q*	19.23	11	
900	7.13*q*		257*q*		1669	787*q*	47*q*	7*q*
930		11.53	149	7.31*q*	107*q*	181	13.37	11
960		157*q*	7*q*	17.73	61*q*	13		
990	29*q*			11.19	17*q*	269*q*	7*q*	101*q*

CONTENTS.

PAGE

On the homogeneous linear difference equation of the second order with linear coefficients (continued). By E. W. BARNES - - - - - - 65

Determination of successive high primes. By LT.-COL. CUNNINGHAM and H. J. WOODALL - - - - - - - - - - - - - 72

Papers for insertion should be sent to Dr. Glaisher, Trinity College, Cambridge, or to Messrs. Metcalfe and Co. Limited, Printing Office, Trinity Street, Cambridge.

An index to vols. i—xxv has been published. It is issued gratis to subscribers, but may be obtained separately, price Two Shillings.

No. CCCCII.] NEW SERIES. [October, 1904.

Stanford University Library
NOV
30
1904

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XXXIV.—NO. 6.

MACMILLAN AND CO.
London and Cambridge.
1904.

Price—One Shilling.

METCALFE AND CO. LIMITED, CAMBRIDGE.

$30m$	r							
	29	27	23	21	17	11	9	3
1020	23.79	443q	31.41	19	13q			
990	383q	181q	1613	49.11	2819	269q	71q	659q
960	11	67q	7q	31q	2137	73q	7.113	569q
930	13	17.23				7.61q	89q	13q
900		7.19q	121	211q		23q	43	223q
870	7.101	439q	599q	47.79	29q	13	11.41	337q
840	317q	719q	103	37.53	7.11	19q	887q	7.379
810				1171	157q	17	23.61	11.19
780	2027		13q	7.349	41.47	11q	293q	
750		13.13	7.23	29q		89	7.149	17q
720	1913	11q	107	83q	19.43	7.433		
690	17.67	49q	29.37		643q		13.19	769q
660	7q	1087		121	1013	31q	241q	
630	11q	103q		17.43	7.137	1709	37	7.41q
600	151q	947q	19q	13.59	17q	653q	31.47	
570		29q	11q	7.739	1231		1423	71q
540	13		7q	97q	563q	41q	7 11	13.23
510	29		73q		11q	343q		157q
480	19q	7.137	59q	347q	23q	13.37		11q
450	7.43	193	1181	19q		11	2683	61q
420	109	17	2617	1373	7.53q	71q	487q	49.29
390	863q	11.37	13.17	251q		127q		43q
360	1439	13.41	457q	7.859	131q	271q	431q	
330	23	19q	7q	11			7.233	
300	11q	47		449q	67q	7.17	13q	2311
270	31q	7q	101	113q		19	2423	2549
240	7.37q	23.59	11		13		29q	17.19
210		31q	47	13	49.89	23.43	11q	7q
180	17.61	167q	293q	139q	121	29q	1583	67q
150	13		97q	7.953	19q	443q	331q	11.13
120	59q	71	7	17.41		11	49.19	
90	53q		31		17q	7.13		
60	307q	7.11	23	173q	83			
30	7.41q		19	31	37q	797q	17	677q
.			13	11q	7.29q	223q		7q

$30m$	r'							
	1	3	7	9	13	19	21	27
.	11q	13q	941q	23q	421q		97q	37q
30	227q	397q	43	7.601	2089	47	73q	79q
60	19.19	17q	49.11	181q	31q		7.83q	59q
90	251q	53q	17q	19.29		7q	11	1049
120		7.43q	167	89q	11.13			23
150	7q	2791	29.61	13q	2053		271q	11q
180	991q	1543	41		7.23	11.17	229q	7q
210	13q	19	1447	67q			881q	13.13
240	641q	11q		7.37q		31q		17.17
270		29	7.151	61.71	523q	13.19	7q	2609
300	17		89q	11q	101q	7.733	31q	19
330	11.23	7.857	463q		73q		41q	
360	49		13q	17q	79		43q	113q
390	2689	83q	11.37	317q	7.17	97	53q	7q
420	71.83	23	2557	157q	41		11.19	29q
450				49	11q	23q	13.17	239q
480		2083	7.67q	73q	431q	2659	7	11.31
510		383q	19.53	47q	13.61	7.121	79	
540	107q	7q	2213	13q	43q	59q	23q	2347
570	7.719	11.17	449q	103		137q		41q
600	13	97q	17.23		7.47	37q	29.67	7.13q
630	19.31	139q	1163	11.43		1931		
660	11q	89		7.19q	379q	13.29	193q	
690		31.37	7q	23q	2011	17q	7q	101q
720			11	53q		7q		
750	2357	49 83	13q				121q	17
780	7 73q	13.19	1301		11		47	2521
810	17.43		31q		7q	2113		7.11
840	37.89	41	71.79		29q	11.19	13q	
870				7.17	23q	61.67		19.43
900	1103	11q	7.997	179q	13.13	83	7q	73q
930		131q	283q	13		49q	191q	173q
960		7q		11.29	19q	2003	17q	
990	7.11		1999	1453	31q		19.61	13q

Least Factors (>5) *of* $N = 99{,}998{,}000 + (10m + r)$. Tab. VIIa.

[σ denotes 7_2 $\quad$ ε denotes 11].

$10m$	r									
	0	1	2	3	4	5	6	7	8	9
0	φ	$9\mathcal{Q}$	13.31	7.29φ	φ	43φ	244ε	19	11	5923
10	7.157	7559	17φ	37	359φ	13.53	41φ	7.109	6073	11φ
20	23φ		127φ	1049	49φ	421φ	19φ	107φ	13.73	17φ
30	121	7φ	29.61	31.71	431φ	4289	67φ	φ	7.59φ	1277
40	227φ	11.13	79	23	$4\mathcal{Q}$	7.19φ	17.17	4691	43φ	$3\mathcal{Q}$
60	37		7.11φ	3049	13	163φ	$8\mathcal{Q}$	41φ	$2.3\mathcal{Q}$	7.733
60	269φ	29φ	$2\mathcal{Q}$	11.17	19.31	103	7.23φ	13φ	53.83	709φ
70	1579		149φ	49φ	11	φ	97φ	613φ	239φ	331φ
80	7.13	137φ	2357	19.19	$4\mathcal{Q}$	11φ	89φ	7.37φ	571φ	23
90	29φ	43φ	$4\mathcal{Q}$	13.47	7φ	31φ	11φ	17.59	41φ	
100	φ	7.73φ	19φ	67	71φ	1567	13	$\varepsilon 113$	7.277	$9\mathcal{Q}$
110	1583		23φ	347φ	17	7φ	$4\mathcal{Q}$	229φ	11φ	13.29
120	φ	19.53	49φ	263	37φ	193φ	31	$9\mathcal{Q}$	φ	7.11
130	241φ	17φ	13	$3\mathcal{Q}$	43	23	7φ	449φ	$2\mathcal{Q}$	4ε
140	11.19	743φ	$2.3\mathcal{Q}$	$7\mathcal{Q}$	φ	13.13	2579		17.29	127
150	7φ	121	173φ	4049	61.61	$5\mathcal{Q}$	59	7.31φ	13.23	19φ
160	φ	37φ	11φ	$9\mathcal{Q}$	7.547	17φ	$2.3\mathcal{Q}$	157φ	103φ	$3\mathcal{Q}$
170	67φ	49.13	401φ	11.97	53.73	71.89	φ	29.43	7.19φ	313φ
180	41φ	23	17φ	593φ	11.13	7.197	257	47	31	307
190	φ		7φ	353φ	6449	11	φ	13.19	37φ	7.17φ
200	79φ	8387	$2.3\mathcal{Q}$	6659	23φ	φ	7.11		φ	4337
210	13	3823	619φ	$7\mathcal{Q}$	$2.3\mathcal{Q}$	59.61	17.19	11φ	137	31φ
220	343	4ε	$2\mathcal{Q}$	13φ	φ	φ	φ	7.23	11	1163
230	$2.5\mathcal{Q}$		φ	17φ	7.47	19.29	13	67φ	2153	$11\mathcal{Q}$
240	151φ	7.107	$2\mathcal{Q}$	467φ	φ	3163	71φ	73φ	7φ	13
250	11.17		$4\mathcal{Q}$	φ	19φ	7φ	φ		599φ	3119
260	φ	$11\mathcal{Q}$	7.13	43	29.29	φ	$2\mathcal{Q}$	17	139φ	49φ
270	97φ	103φ	12ε	19.23	59φ	13φ	7.61	463φ	1291	79φ
280	53φ	31.47	101φ	7.11φ	17	389φ	$2.3\mathcal{Q}$	2963	13φ	$9\mathcal{Q}$
290	7.953		19φ	29	11φ	1013	23	$7\mathcal{Q}$	φ	
300	φ	13.17	$2\mathcal{Q}$	4ε	7.67φ	11φ	43φ	φ	1117	37φ
310	φ	7.19φ	31φ	2663	13.13	$5\mathcal{Q}$	11	71.71	49.17	23
320	73φ	301ε	29φ		157φ	7.173	131φ	11.13	47φ	223
330	19φ	$3\mathcal{Q}$	7φ	53.59	φ	17.17	569φ	61	11φ	$7\mathcal{Q}$
340	13φ	2089	23φ	31φ	4ε	397φ	7.37	337φ	107φ	11.19
350	179φ	29	17φ	7.13	$2\mathcal{Q}$	137φ	631φ		79φ	827φ
360	7.11φ	φ	2273		φ	23φ	13	49.97	19φ	17φ
370	149φ	11.67	773φ	359φ	7.3ε	47φ	φ		673φ	13
380	29φ	7.163	11	37	φ	4ε	17.53	19	7.23φ	2543
390	443φ	151φ	13.43	121	607φ	7φ	$4\mathcal{Q}$	$9\mathcal{Q}$	61φ	863φ
400	83φ	1459	7φ	17	11φ	13.31	19φ	139φ	$8\mathcal{Q}$	7.29φ
410	331φ	23φ	503φ		$2\mathcal{Q}$	11φ	49φ		13φ	8231
420	17.37	$3\mathcal{Q}$	47φ	$\sigma 3623$	617φ	19φ	11.41	1847	$4\mathcal{Q}$	
430	7φ	13	191φ	$9\mathcal{Q}$	23φ	43	31φ	7.11	29.67	53φ
440	541φ		89φ	257φ	7.13	φ	113		11φ	
450	φ	7.59φ	1427	109	17φ	107	1217	13.23	7.27ε	11.61
460	347φ	653φ	$2\mathcal{Q}$	19φ	97φ	49	73φ	29.31	$4\mathcal{Q}$	47φ
470	11.13	17φ	7φ		233φ	φ	$4\mathcal{Q}$	103φ	43φ	$\sigma 1423$
480	23φ	11	19φ	13.13	101φ	$5\mathcal{Q}$	7φ	$9\mathcal{Q}$	17φ	929φ
490	φ		11.53	7.439	37	199φ	13.29		31	3187

Least Factors (>5) *of* $N = 99{,}998{,}000 + (10m + r)$. Tab. VII*b*.

[σ denotes 7, ε denotes 11].

$10m$	r									
	0	1	2	3	4	5	6	7	8	9
600	7*q*	19	2.3*Q*	11.23	8*Q*	17.67	2*Q*	σ1213	41*q*	13*q*
610	59*q*	193*q*	*q*		49.11	2539	47.79	3*Q*	181*q*	149*q*
620	19.61	7.43*q*	13.17	*q*	739*q*	11.29	23*q*	7481	7*q*	31
630	71	37.89	1033	491*q*	2*Q*	7.13*q*	11	1153	2.3*Q*	17.19
640	419*q*	2417	7.151		163*q*	53*q*	139	11*q*	13*q*	7.23
650	*q*		223*q*	3*Q*	29*q*	1039	7.17		11.19	113*q*
660	31*q*	13.97	107	343	43	1087	83*q*	1451	37*q*	11.59
670	7	3*Q*	23.67	17	13*q*	2293*q*	2203	7.19*q*	2*Q*	197*q*
680	11	61	2*Q*	29*q*	7*q*	101*q*	449*q*	13	131*q*	3*Q*
690	17.41	7.11	*q*	1259	2*Q*	23.79	19*q*	211*q*	7.53	4421
600	13*q*	71	11		*q*	7.37	887*q*	17.43	*q*	
610	47*q*		49.29	11.13	2*Q*	19	283*q*	4211	23*q*	7.383
620	89*q*	349	31*q*	191*q*	11.17	*q*	7.13*q*	281*q*	59*q*	137
630	2.5*Q*	41*q*	487*q*	7*Q*	19*q*	121*q*	2411	659*q*	157*q*	13.67
640	7*q*	17.23	37.61	3*Q*	4*Q*	5*Q*	11	7*Q*	911*q*	83
650	43*q*	53	13.13	19.31	7.751	*q*	521*q*	11.47	17.97	571*q*
660	*q*	49*q*	277*q*		23*q*	13*q*	1319	9*Q*	7.11	107*q*
670	29*q*	109	19.41	3461	79	7.17*q*	*q*	2039	13*q*	11.37
680	*q*	3467	7*q*	103*q*	31	73.73	101*q*	23.59	*q*	7.3731
690	11	13.19	17	43	199*q*	5*Q*	7*q*	3*Q*	2*Q*	29
700	257*q*	11*Q*	2*Q*	7.61*q*	13.47	1721	67*q*	163	179*q*	17.89
710	49.19	547*q*	11*q*	41	5839	31*q*	37	7.13	2.3*Q*	131*q*
720	*q*	227*q*	2*Q*	11*Q*	7.661	*q*	17*q*	3*Q*	29*q*	19
730	13*q*	7*Q*	83	23	11*q*	1303	43*q*	461*q*	7	1327
740	463*q*	331*q*	397*q*	13.17	1543	7.11*q*	31.59	2729	19*q*	
750	*q*	47*q*	7.433	37.79	41*q*	97*q*	121	29.53	73*q*	499
760	17*q*	5519	2389	1301	61*q*	167*q*	7.137	11.19	*q*	13*q*
770	2.5*Q*	379*q*	*q*	7.67*q*	2*Q*	223*q*	107	17.31	11	23.43
780	7.109	3*Q*	13		127*q*	113*q*	19.29	7.101	4*Q*	1607
790	37*q*	9719	733*q*	797*q*	7.17*q*	13.41	1531		47.89	3*Q*
800	11*q*	7*Q*	23*q*	307*q*	229*q*	19.19	2*Q*	347*q*	49.13	2143
810	53*q*	11.17	4*Q*	1181	71	7.29	*q*	149*q*	2*Q*	
820	*q*	13.13	7.11	3*Q*	19	23.61	2017	37*q*	17*q*	7*q*
830	2.5*Q*	73	79*q*	1443	13*q*	*q*	7.41*q*	991*q*	*q*	31*q*
840	67*q*	3*Q*	1823	7.19*q*	11.29	17.47	2*Q*	13*q*	23*q*	293*q*
850	7.131		97.97	241*q*	2*Q*	11*q*	*q*	49	3209	3*Q*
860	13*q*	3517	17.19	53	7.37	43*q*	11*q*	269	*q*	
870	31	7.23*q*	809*q*	13.29	*q*	*q*	4*Q*	121	7	17*q*
880	181*q*	19	2*Q*	107*q*	743*q*	7.71*q*	13.61	89	11	103
890	2.5*Q*		7.47*q*	199*q*	23*q*	*q*	17	193*q*	83	7.11
900	19*q*	31.37	29	137*q*	73*q*	569*q*	343*q*	67*q*	43	
910	11*q*	79	13	7.17	479*q*	503*q*	53*q*	23	41	19*q*
920	7*q*	11*Q*	2.3*Q*	59*q*	4*Q*	13*q*	2*Q*	7*Q*	*q*	
930	17*q*	29*q*	11.31		7.563	4051	8*Q*	1559	13.19	47
940	23	7*Q*	2*Q*	11	*q*	5*Q*	*q*	17.61	7.227	97*q*
950	*q*	13.43	157*q*		11	49*q*	71*q*	19	2.3*Q*	41
960	29*q*	3*Q*	7*q*	23.31	13.17	11*q*	149*q*	2477	8*Q*	7.53*q*
970	349*q*		587*q*	197*q*	67	37*q*	7.11	13.73	2*Q*	3*Q*
980	*q*	17.83	59*q*	7*Q*	8*Q*	*q*	23.23	1631	1439	29
990	7.143	9*Q*	103*q*	677*q*	31.43	19	4159	7*q*	121	

Least Factors (> 5) *of* $N = 99{,}999{,}000 + (10m + r)$. Tab. VIIc.

[σ denotes 7, ε denotes 11].

10*m*	*r*									
	0	1	2	3	4	5	6	7	8	9
0	41*q*	607*q*	383*q*	13*q*	49*q*	191*q*	2·3*Q*	4523	61*q*	11.23
10	2.5*Q*	7.113	37*q*	2621	19*q*	17*q*	13*q*	173*q*	7.29*q*	211*q*
20	11*q*	337*q*	53*q*	6701	491*q*	7·31*q*	647*q*	71*q*	4*Q*	13*q*
30	*q*	11*Q*	7.17	19.47	239*q*	449*q*	*q*	43*q*	127*q*	7
40	137*q*	41.59	11.13		4*Q*	857*q*	7·97*q*	29*q*	877*q*	17.37
50	73*q*	673*q*	19	49.11	1231	13.23	31*q*	3*Q*	2*Q*	367*q*
60	7*q*	181*q*	373*q*	389*q*	11.83	89*q*	17	7*Q*	13*q*	61.79
70	2.5*Q*	19*q*	331*q*		7*q*	11.53	29*q*		23*q*	
80	43.47	7·13*q*	41*q*	17	*q*	5*Q*	11.37	31*q*	7*q*	
90	19	359*q*	199*q*	3*Q*	13	7.103	*q*	ε107	71*q*	167*q*
100	17.59	23	49		*q*	29.29	2*Q*	13	11.67	7.19
110	307*q*	3*Q*	131*q*		653*q*	149*q*	7*q*	17*q*	31	121
120	13*q*	1399	2*Q*	7.37	23	*q*	*q*	47*q*	19.53	941*q*
130	7.11		*q*	13*q*	17.29	*q*	269*q*	σ3727	2.3*Q*	397*q*
140	*q*	11*q*	293*q*	97*q*	7*q*	773*q*	13	19.23	79*q*	31*q*
150	379*q*	49.17	11*q*	251*q*	89	5*Q*	2557		7.509	13.13
160	37*q*	5081	*q*	11.29	41*q*	7.127	19.19		17*q*	71
170	23	*q*	7.13*q*	233*q*	11.47	67	8*Q*	137	2*Q*	7*Q*
180	31*q*	53*q*	283*q*	3*Q*	*q*	11.13	7.827		601*q*	281*q*
190	173*q*	61	29*q*	7.23*q*	2*Q*	*q*	11.73	37*q*	13	3617
200	49*q*	3*Q*	17.17	463*q*	19	41	2*Q*	7.11*q*	8*Q*	43*q*
210	*q*	13.31	4*Q*	3*Q*	7.257	5*Q*	23		11.59	17*q*
220	*q*	7.29	709*q*	19	13*q*	*q*	2*Q*	79*q*	7*q*	ε1097
230	83	3*Q*	*q*	9433	37.53	7.317	17*q*	13	2*Q*	23
240	121	139*q*	7.19	89	4*Q*	179*q*	41*q*		*q*	49.49
250	13.29	ε311	43.61	17*q*	2*Q*	*q*	7*q*		547*q*	
260	*q*	19*q*	11.23	7.13	149*q*	2029	157	2339	47*q*	73*q*
270	7.17*q*	37	8*Q*	11.31	2*Q*	*q*	13	7.59*q*	2*Q*	29*q*
280	19*q*	2311	2.3*Q*	193*q*	7.11*q*	23*q*	2*Q*	17.41	421*q*	13*q*
290	2.5*Q*	7.199	101	577*q*	2.3*Q*	11.43	659*q*	151*q*	49*q*	19
300	*q*	103*q*	13*q*	613*q*	17.31	7*q*	11.79		23.29	67*q*
310	2.5*Q*	71	7*q*	61.83	137*q*	13.47	1277	11*Q*	19*q*	7.349
320	*q*	17*q*	3001		557*q*	109*q*	7*q*	431*Q*	11.13	
330	*q*	23	89*q*	7*q*	2*Q*	31	59*q*	19.29	17.43	11*q*
340	7·53*q*	13*q*	73*q*		*q*	37*q*	2*Q*	49*q*	*q*	1597
350	11	9*Q*	8*Q*		7.13	17*q*	19*q*	3*Q*	337*q*	163*q*
360	643	7.121	47*q*	173*q*	4*Q*	5*Q*	29.31	13	7.787	41
370	2.5*Q*	439*q*	11.17		61	7.19*q*	67*q*	23*q*	419*q*	2333
380	13	43*q*	7.37	11*Q*	*q*	79	2*Q*	191*q*	4*Q*	7 17*q*
390	*q*	3821	1163	13.53	11.19	29.59	49.83	31*q*	2*Q*	3*Q*
400	23*q*		227*q*	7.331	103	11	13.17	6781	*q*	47*q*
410	7·41*q*	3*Q*	947*q*	19	*q*	73*q*	11*q*	7.307	107	13.37
420	*q*	89*q*	2*Q*	17.23	7.29	1117	367*q*	11*Q*	31*q*	3*Q*
430	2.5*Q*	7.313	13.19	167*q*	97	61	4*Q*		7.11	
440	17*q*	211*q*	821*q*	67*q*	223*q*	49.13	23.53	691*q*	151*q*	11*Q*
450	359*q*	19.41	7.389	29.71	59*q*	5*Q*	37.47	17*q*	13*q*	7·31*q*
460	11*q*	3877	2.3*Q*	113*q*	79*q*	283*q*	7*q*	43.43	*q*	23*q*
470	19	11.13	*q*	7*Q*	17	1301	193*q*	9*Q*	2*Q*	83*q*
480	7*q*		121	3*Q*	13	5*Q*	*q*	7.709	73*q*	19*q*
490	31	17.17	23.41	11.37	49	*q*	61*q*	13.13	2.3*Q*	53*q*

Least Factors (>5) *of* $N = 99{,}999{,}000 + (10m + r)$. Tab. VII*d*.

[σ denotes 7, ε denotes 11].

	r									
$10m$	0	1	2	3	4	5	6	7	8	9
500	q	7·347	$2Q$	$47q$	$11q$	$131q$	$2Q$	$103q$	7·17	
510	13·43	$29q$	$239q$	$59q$	$2Q$	7·11	1063		$2Q$	$139q$
520	q	$31q$	7·163	$13q$	$71q$	17	$11q$	19·19	q	7·379
530	$37q$	$97q$	1973	$41q$	$2{\cdot}3Q$	3067	7·13	$11q$	$23q$	
540	$29q$		17	49·79	$811q$	$5Q$	19		11	$13q$
550	$7{\cdot}47q$		31·53	$43q$	$2Q$	1783	$4Q$	$7{\cdot}61q$	1741	11·17
560	q	23·73	13·83		7·197	19	$2Q$	$37q$	q	29
570	$11q$	7·839	$59q$	$3Q$	$41q$	$13q$	17	67	7·191	$3Q$
580	$157q$	$11Q$	$2{\cdot}3Q$	$31q$	19·23	$7q$	1283		13	
590	$797q$	1381	343	$17q$	1039	71	$43q$	$47q$	$29q$	7·89
600	$167q$	$13q$	$2Q$	121	37	53	7·487	23	$607q$	$3Q$
610	17		q	$7Q$	11·13	$41q$	q		61	2381
620	$7q$	$9Q$	19·79		q	$11q$	$859q$	7·13	$97q$	$227q$
630	$23q$	$59q$	$107q$	2423	7·73q	$5Q$	11	2267	$2Q$	43
640	$13q$	49·19	$2{\cdot}3Q$		17·47	31·83	1193	$\varepsilon 1493$	$7q$	1579
650	q	$3Q$	109	13·23	$2{\cdot}3Q$	7·251	29·41	q	11·53	$461q$
660	$19q$	$17q$	$7q$	1093	q	3907	13·13	223	$349q$	7·11
670	$677q$	3257	1567	127	$269q$	q	7·23		17·37	13·19
680	$11q$	$809q$	43	$7Q$	q	29	$2Q$	2141	$89q$	113
690	49·59	11·47	13	1583	$2Q$	17	$101q$	$7{\cdot}41q$	19	$23q$
700	$757q$	79	$11q$		$7q$	$13q$	$2Q$	31·73	q	173
710	$2{\cdot}5Q$	7·53	$17q$	$\varepsilon 103$	$29q$	$37q$	$4Q$	$19q$	7·13	$443q$
720	$229q$		$23q$	$3Q$	121	7·43	$2{\cdot}3Q$	7951	83	$17q$
730	$2{\cdot}5Q$	$13q$	$7q$	1789	$331q$	$11q$	$19q$	71	31·31	49
740	$61q$	$3Q$	1699	$29q$	$13q$	$23q$	7·11	$9Q$	$283q$	59
750	$151q$	239	$37q$	7·701	$2Q$	$19q$	q	11·13	$431q$	$3Q$
760	$7q$	109	3079	$17q$	$53q$	q	$2Q$	7·131	11·23	31
770	13	$3Q$	$29q$		7·19q	1997	$401q$	$89q$	$67q$	11·41
780	17·17	$\sigma 1123$	$179q$	13	2909	47	$2{\cdot}3Q$		$49q$	37
790	$11q$	$23q$	q	19	$2Q$	$7q$	$13q$	17	1811	2063
800	31	11·29	7·113	1021	1669	$547q$	$2Q$	$3Q$	59·71	7·13
810	$439q$	43·83	11·19	$271q$	17·23	$5Q$	7·103	53	1223	1453
820	$41q$		13·97	7·11q	q	q	37		$991q$	7907
830	$7{\cdot}29q$	17·19	$47q$	3361	$11q$	13·13	$4Q$	49·23	2113	
840	q	$389q$	$2Q$	$3Q$	$7q$	121	$107q$		13·17	$3Q$
850	$19q$	$7Q$	1259	$73q$	$43q$	q	11	3433	7·977	29·79
860	23	13·41	31·61	37	q	7·17q	$89q$	11·59	$4Q$	19
870	53	7451	7·233	$311q$	$13q$	q	q	1307	$11q$	7·47
880	q	$557q$	17	$23q$	$4Q$	1753	$49q$	13	19·19	$\varepsilon 1933$
890	$223q$	$9Q$	$4Q$	7·31q	83	2297	$8Q$	$43q$	101	$17q$
900	7·11	$151q$	41	2741	q	$5Q$	23	7·19q	$379q$	$9Q$
910	$2{\cdot}5Q$	$11Q$	$67q$	$13q$	$7q$	$113q$	17	$29q$	$773q$	97
920	q	7·503	$11q$	53·61	$31q$	$877q$	13·19	$127q$	$7q$	$23q$
930	q		$4Q$	11·17	$167q$	$343q$	$139q$	$37q$	$79q$	$13q$
940	$43q$		$7q$	41	$11q$	$19q$	29·29	4013	q	7·229
950	17·71	3331	13·23	$107q$	q	11·31	$7q$	$3Q$	197	
960	$191q$	179	$593q$	$7q$	19	$13q$	121	$17q$	q	1091
970	$7q$		1009	$47q$	37·37	23·29	$53q$	7·11	$13q$	67
980	q	q	$2Q$	19·43	49·17	59	$31q$	$3Q$	$11q$	
990	$239q$	7·13	$8Q$	$3Q$	1181	$5Q$	1667	1297	7·23q	11·73

Least Factors (>5) *of* $100{,}000{,}000 + (10m + r)$. Tab. VIIIc.

[σ denotes 7, ϵ denotes 11].

10m	r									
	0	1	2	3	4	5	6	7	8	9
1000	$11q$	$3Q$	1511	5431	$17q$	13	$7q$	$47q$	$389q$	19 31
1010	23.37	$\cdot 941$	3271	49	$2Q$	43.71	$263q$	$353q$	13.13	29
1020	7.193	17.61	11	89	$577q$	q	$2Q$	$\sigma 2731$	$19q$	
1030	$2.5Q$	13	1733	11.23	$7q$	$179q$	$53q$	$233q$	$17q$	5^813
1040	$31q$	$\sigma 1223$	$461q$	$337q$	11.13	113	$2.3Q$	19.37	7.29	$3Q$
1050	$41q$	67	103		47	7.121	23.83	$13q$	43	
1060	1091	q	$49q$	5147	$349q$	$5Q$	11.19	$487q$	$4Q$	7.109
1070	13	$31q$	$17q$	$653q$	$2Q$	q	$7q$	11.29	$2Q$	$23q$
1080	q		61	7.13	37	$19q$	$71q$		$11q$	17.53
1090	$7q$	$41q$	$4Q$	$173q$	73	$137q$	$13q$	7.499	3673	$11Q$
1100	q	43.47	23,23	$167q$	$7.19q$	$5Q$	17.29		$4Q$	13
1110	11	49	$89q$	$743q$	$2Q$	q	$4Q$	$311q$	$7.67q$	
1120	q	11.37	$13q$	17.19	$397q$	$7.23q$	$2Q$	$439q$	$641q$	$211q$
1130	$107q$		7.11	$31q$	$2Q$	13.29	q	$317q$	$2Q$	$7.83q$
1140	17	2797	19.53	11.61	$43q$	$541q$	$7q$		13.23	$953q$
1150	$157q$	2389	$163q$	$7Q$	$11q$	103	$281q$	17.71	37	
1160	$49q$	13.19	4591	$283q$	29.31	11.59	$811q$	$7.73q$	$271q$	q
1170	1049	$23q$	q	41	7.13	1129	$121q$		$109q$	1663
1180	19.97	7.569	1151		$797q$	67	$563q$	11.13	$7q$	1187
1190	q	$17q$	3167	29	23	7.31	q	127	$11q$	$19q$
1200	$13q$	89	$7.79q$		$61q$	q	2819		17	49.11
1210	$2.5Q$	$101q$	$4Q$	13	41	$5Q$	$7q$	$23q$	19	
1220	$11q$	2689	29.83	$7q$	$59q$	17.17	13.31		71	3761
1230	$7.43q$	$11Q$	37	$181q$	$2Q$	q	$347q$	7.19	$2.3Q$	13
1240	23.73	1949	11.17	6653	$7q$	$359q$	6599	$3Q$	$53q$	$839q$
1250	2963	$7.29q$	13.67	11	$2Q$	$41q$	$19q$	$31q$	49	17
1260	$151q$	367	$2.3Q$	23	$11q$	7.13	$173q$	$887q$	q	$37q$
1270	167	$113q$	7	43	$2.3Q$	11.19	17	$97q$	$13q$	7.263
1280	$29q$	79	$2Q$	$59q$	$4Q$	$907q$	7.11	$109q$	31	47
1290	$89q$	$13q$	q	$7.17q$	19	q	$41q$	121	$2.3Q$	71
1300	7.373	$53q$	2801		$13q$	$83q$	37	343	$11q$	23.29
1310	$17q$	5179	$101q$	19.73	$7q$	$163q$	43.43	$13q$	$409q$	11.31
1320	$587q$	$\sigma 2281$	q	$229q$	$127q$	$937q$	61	$17q$	$7q$	
1330	11.13	q	19.19		1013	$7q$	$47q$	41	29	$131q$
1340	$211q$	$\epsilon 1303$	$7.59q$	13.37	17	$5Q$	$2.3Q$		$971q$	$7q$
1350	31	$19q$	$11q$	7247	$53q$	$23q$	49.13		q	$43q$
1360	$79q$	17	2287	$7.11q$	q	1301	$2Q$	$29q$	$661q$	13
1370	7.19	3631	1061	$3Q$	11.97	q	q	$\sigma 2207$	17.23	$89q$
1380	37	31	$13q$	$47q$	7.113	$11q$	67.73	$61q$	$83q$	19
1390	$2.5Q$	$7q$	q	179	q	13.17	11.29	$389q$	$7q$	
1400	q	23.59	$43q$	$9Q$	$4Q$	$49q$	$193q$	11.53	13.19	4261
1410	$2.5Q$	$151q$	7.17	101	$181q$	q	$8Q$	$37q$	$121q$	$7.41q$
1420	1429	$13q$	$2Q$	1097	23	$29q$	$7q$	$19q$	$311q$	11.17
1430	47.47		$277q$	$7q$	$13q$	$5Q$	q	167	$2Q$	79
1440	$7.11q$	71	$2.3Q$	$31q$	2477	43	17.19	7.13	$61q$	
1450	q	11	$4Q$	67.67	49.29	$853q$	q	$9Q$	$2Q$	$73q$
1460	13.41	$7Q$	$11q$	$17q$	103	19	q		$7.89q$	1451
1470	23	83.97	q	11.13	$31q$	$7q$	2729	$47q$	$163q$	$491q$
1480	17	3863	7.43^1	$29q$	11.19	$5Q$	13	$3Q$	43	$7Q$
1490	q	37	$4Q$	$23q$	$2Q$	$11q$	$7q$	17	3557	13

Least Factors (>5) *of* 100,000,000 + (10*m* + *r*). Tab. VIII*d*.

[σ denotes 7, ε denotes 11].

10*m*	*r*									
	0	1	2	3	4	5	6	7	8	9
1500	*q*	41*q*	*q*	49.19	241*q*	31	11		*q*	61
1510	7*q*	631*q*	13.29	53*q*	17.17	5*Q*	23	7.11*q*	79	59*q*
1520	67*q*	397*q*	19	3539	7.47*q*	13.13	2.3*Q*	257*q*	11.37	769*q*
1530	2791	7.17	73*q*		197*q*	*q*	31*q*		7.13*q*	121
1540	*q*	19.29	41	2897	*q*	7*q*	2*Q*	*q*	17*q*	2447
1550	11*q*	13	49	3*Q*	83	5*Q*	*q*	89	107*q*	7*q*
1560	19*q*	11*Q*	23	1481	13*q*	17.37	7.53	31	97*q*	
1570	29.61	47*q*	11	7.983	43*q*	*q*	8*Q*	13*q*	59	19*q*
1580	7*q*		17*q*	11.41	*q*	23*q*	2.3*Q*	7.67*q*	4903	3797
1590	13*q*		601*q*	6661	7.11*q*	181*q*	499*q*	79*q*	19.31	17.29
1600	*q*	49	37*q*	13	139	11.73	2339	1657	7.23*q*	1811
1610	113		173*q*	5507	109	7.101	11.13	19.43	47*q*	53*q*
1620	*q*	157*q*	7.313		41*q*	*q*	2*Q*	11*Q*	29.29	7.13
1630	2053	23.23	*q*	17*q*	2621	5*Q*	7.19*q*	59.83	11	37*q*
1640	*q*	163*q*	13*q*	7.137	4*Q*	5*Q*	89*q*	5737	149*q*	11*q*
1650	49.49		*q*		23.67	13.19	1483	7.29*q*	751*q*	1117
1660	121	3*Q*	1601	239	7*q*	41.47	3347	17	13*q*	5077
1670	103*q*	7.11*q*	53	373*q*	19	*q*	37.79	23*q*	7.73*q*	191
1680	*q*	13*q*	11*q*	383*q*	17*q*	7*q*	29		*q*	
1690	2.5*Q*	31*q*	7.61*q*	11.19	13.13	5*Q*	59*q*	3*Q*	2*Q*	49*q*
1700	23*q*	17	2*Q*	43	11*q*	127*q*	7.41*q*	13*q*	4*Q*	277*q*
1710	271*q*	3671	19.47	7.37	2*Q*	11.29	101*q*	337*q*	17	
1720	7.113	67*q*	31*q*	23	733*q*	53.71	11	7.409	*q*	283
1730	*q*	19	131*q*	13*q*	7*q*	17.89	233*q*	11*Q*	6841	311*q*
1740	*q*	7.593	2.3*Q*	139	29*q*	241*q*	13.23	41*q*	49.11	563*q*
1750	19.37	73	17*q*	31.61	*q*	7.59	4*Q*	3*Q*	727*q*	11.13
1760	*q*		7.97	*q*	4373	5*Q*	2.3*Q*		8*Q*	7.17
1770	11*q*	167	13	29	1747	*q*	7.181		53	293*q*
1780	137*q*	121	521*q*	7.433	31	13	17	37*q*	19.41	43*q*
1790	7*q*		11.23	3*Q*	2*Q*	5*Q*	71*q*	49	13*q*	9*Q*
1800	*q*		29*q*	11.17	7.163	263*q*	47*q*	19*q*	*q*	
1810	991*q*	7.13*q*	977*q*		11.59	23.31	8*Q*	101*q*	7*q*	
1820	17*q*		1933	3*Q*	13.37	7.11*q*	19		2141	41*q*
1830	1543	29.53	7.43	4943	79*q*	*q*	11	13.17	23*q*	7*Q*
1840	*q*	3*Q*	2*Q*		*q*	19	49.31	11*q*	8*Q*	1093
1850	13*q*	863*q*	599*q*	7.47*q*	17*q*	67*q*	701*q*	1109	11	97*q*
1860	7.29*q*	23.37	2003	13.13	19	*q*	1439	7.71*q*	*q*	11*Q*
1870	41	17*q*	*q*	59	7*q*	43.61	13	31*q*	199*q*	107*q*
1880	11*q*	7*Q*	139*q*	19*q*	23.53	5*Q*	83*q*		7.17*q*	13.29
1890	2.5*Q*	ε1549	419*q*		2*Q*	49*q*	1831	73	37*q*	227*q*
1900	47*q*	3*Q*	7.121		919*q*	17	2*Q*	23	31*q*	σ1289
1910	*q*	41*q*	2243	11.79	251*q*	13*q*	7*q*	137*q*	29.43	1399
1920	*q*	19*q*	17.67	7*Q*	11*q*	*q*	619*q*		13	
1930	7.23	3*Q*	59	503*q*	443*q*	11.37	61*q*	7.53*q*	71	17.31
1940	19*q*	13	2*Q*	9*Q*	49*q*	5*Q*	11	29.47	4*Q*	113*q*
1950	*q*	σ2113	41*q*	23*q*	13*q*	*q*	17.97	ε181	7.173	19
1960	*q*	43*q*	2*Q*	359*q*	461*q*	7*q*	2*Q*	13	11*q*	83
1970	31.73	757*q*	7.37*q*	17	1999	*q*	23.29	811*q*	19*q*	7.11
1980	13*q*	271	2.3*Q*		*q*	193*q*	7.107		*q*	67*q*
1990	11.17	59*q*	79*q*	343	47	367*q*	1153	19 61	2*Q*	23

ON SYNTHETIC DIVISION.

By *H. J. Woodall, A.R.C.Sc.*

[The Author desires to acknowledge the help of Lt. Col. A. Cunningham, R.E., in preparing this paper].

1. A TABLE of Residues (R)* of powers of 10 gives, by inspection, the *Remainder* (R_n) when any power (say 10^n) is divided by any modulus (m) included in the Table; (m is usually prime). Also, although the Table does not give it explicitly, it is possible (when the modulus is prime to 10) by a very simple calculation to obtain the *quotient* (Q_n) of $(10^n - R_n) \div m$, *without any actual division.*

We have, by definition,

$$10^n = m Q_n + R_n \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where R_n is the Residue tabulated for 10^n to the modulus m. Now, from (1),

$$m Q_n + R_n \equiv 0 \pmod{10},$$

whence also

$$\mu q_n + r_n \equiv 0, \text{ and } \mu q_n \equiv -r_n \pmod{10} \ldots\ldots (2),$$

where μ, q_n, r_n are the *units digits* of m, Q_n, R_n, respectively.

2. Result (2) suffices to give q_n from r_n (given by the Tables) by the following very simple Rules. Noting that $\mu = 1, 3, 7, 9$ only (because m is prime to 10), and that q_n, r_n are each < 10,

$\mu = 1$ gives $q_n \equiv -r_n \pmod{10}$; whence $q_n = 10 - r_n \ldots\ldots\ldots\ldots (3a)$.

$\mu = 3$ gives $3q_n \equiv -r_n$; whence $9q_n = (10 - 1)\, q_n \equiv -3r_n \pmod{10}$,

therefore $q_n \equiv +3r_n \pmod{10}$; and $q_n =$ *units digit* of $3r_n \ldots (3b)$.

$\mu = 7$ gives $7q_n \equiv -r_n$; whence $49q_n = (50 - 1)\, q_n \equiv -7r_n \pmod{10}$,

therefore $q_n \equiv +7r_n \pmod{10}$; and $q_n =$ *units digit* of $7r_n \ldots (3c)$.

$\mu = 9$ gives $9q_n = (10 - 1)\, q_n \equiv -r_n \pmod{10}$; whence $q_n = r_n \ldots\ldots (3d)$.

The trifling arithmetical operations for finding q_n by these four rules ($3a$, b, c, d) can be performed mentally.

---

* Jacobi's *Canon Arithmeticus* gives a complete Table of Residues of 10^n (mod. m) for all prime values of $m < 1000$ of which 10 is a primitive root. The author has supplied (at present only in MS.) the deficiency—*i.e.* for values of $m < 1000$ of which 10 is *not* a primitive root.

Also a *Denary Canon* has been prepared (at present only in MS.) by the author and by Lt.-Col. A. Cunningham, R.E. (working independently) showing the residues of 10^n up to $n = 12$ for all moduli (m) which are *primes* or *powers of primes* and $< 10{,}000$.

It will now be shown that the digits $q_1, q_2, \ldots q_n$ found by the above Rules from the end digits $r_1, r_2, \ldots r_n$ of the tabulated Residues are the *required digits* of the quotient Q_n. Since each q_n is (by definition) the units digit of the corresponding Q_n, and Q_1 contains only one digit, we have

$$Q_1 = q_1; \quad Q_2 = 10Q_1 + q_2 = 10q_1 + q_2;$$
$$Q_3 = 10Q_2 + q_3 = 100q_1 + 10q_2 + q_3:$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$Q_n = 10^{n-1}q_1 + 10^{n-2}q_2 + \ldots + 10^2 q_{n-2} + 10q_{n-1} + q_n \ldots (4).$$

This last Result proves the proposition, and may be read *arithmetically*

$$Q_n = q_1, q_2, q_3, \cdots, q_{n-2}, q_{n-1}, q_n,$$

an example will make all this clear.

Ex. Take $m = 353$. The successive Residues (R_n) of the powers of 10 (taken from the Table) are

$n =$	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,
$R_n =$	10,	100,	294,	116,	101,	304,	216,	42,	67,	317,
$q_n =$	0,	0,	2,	8,	3,	2,	8,	6,	1,	1.

The successive digits q_n required for the Quotient Q_n were obtained by simply writing down the units digit of $3r_n$ (where r_n is the units digit of R_n), as in the Rule (3*b*). Hence *any* quotient Q_n may be read at sight, *e.g.*

$$(10^5 - 101) \div 353 = 283,$$
$$(10^{10} - 317) \div 353 = 28328611.$$

The ease and rapidity* with which such quotients may be obtained *without actual division* will now be obvious.

3. Result (2) may also be used for finding the *units digit* (r_n) of the Residue R_n when the *units digit* (q_n) of the Quotient Q_n is given; thus,

$$r_n \equiv -\mu q_n \pmod{10} = 10 - \text{units digit of } \mu q_n \ldots (5),$$

or, again, the actual Residue R_n may be found when the quotient Q_n is *given*; thus

$$R_n = 10^n - mQ_n \ldots\ldots\ldots\ldots\ldots (5a).$$

It is, however, unnecessary to use all the digits of which Q_n consists, because, as (5*a*) shows, if R_n be $< 10^k$, then the product mQ_n will consist of a row of $(n-k)$ 9's followed by the digits in order of $10^k - R_n$, altogether n digits.

[We assume, as usual, that $R_n < m$].

---

* Many of the divisions required in the preceding Paper on *Determination of Successive High Primes* (in *Messenger of Mathematics*, Vol XXXIV., 1904), were effected by the author *by the above process*, thus showing it to be a *thoroughly practical and effective process* for the purpose.

Hence, if m contains k figures, the Residue R_n of $10^n \div m$ is precisely the remainder when the last k figures of the product $m.Q_n$ are taken from 10^k, or more shortly when the last k figures of the product $m.Q_n'$ are taken from 10^k, where Q_n' is the number denoted by the last k figures of Q_n, *i.e.*

$$R_n = 10^k - \text{last } k \text{ figures of } (m \times Q_n') \ldots\ldots\ldots(6).$$

Note (1).—The given quotient should be *of same order* (n) as the required Residue (R_n). If the given quotient be Q_{n+r} of higher order ($n+r$) than the required (R_n), then the quotient Q_n, required (for finding R_n) is found simply by leaving out the extra r figures on the right of Q_{n+r}.

Ex. Let $m = 353$, $Q_{10} = 28328611$, find R_8.

Here $Q_8 = 283286$, and $Q_8' = 286$.

Then $R_8 \equiv 10^8 - 353 \times 286 \pmod{10^8} \equiv 10^8 - 958 = 42$.

In case the order (n) of the given Q_n be less than that of the required R_t, we must first find the R_n (of the same order as the Q_n given).

Then $R_t = \text{Residue of } 10^{t-n}.R_n \pmod{m} \ldots\ldots\ldots\ldots\ldots(7)$

will give the required result.

Note (2).—When the base is 10, the *order* (n) of a given quotient is identically the number of its digits plus the number of digits in the divisor, less one.

4. The principles above developed for the common scale of notation, in which they have their most important application, can be readily extended to scales of any radix whatever.

The simplest case is when the radix $= 2$. Here

$$2^n = m Q_n + R_n \ldots\ldots\ldots\ldots\ldots\ldots(1)_2,$$

where Q_n is the Quotient and R_n is the Residue when 2^n is divided by m.

Now, from $(1)_2$,

$$m Q_n + R_n \equiv 0 \pmod{2},$$

whence also

$$\mu q_n + r_n \equiv 0 \text{ and } \mu q_n \equiv -r_n \pmod{2} \ldots\ldots(2)_2,$$

where μ, q_n, r_n are (as in Art. 1) the units digits of m, Q_n, R_n expressed in the binary scale, and are therefore each 0 or 1, according as m, Q_n, R_n are even or odd. If, as is most usual, we take m to be an odd prime, then $\mu = 1$ and q_n and r_n will be either both $= 0$ or both $= 1$.

Hence, given a Table* of Residues of powers of 2, we may write down (as a number in the *binary* scale of notation) the quotient when any power of 2 (covered by the Table) is divided by a modulus (m).

Ex. Let it be required to find, by Synthesis, the quotient when 2^{29} is divided by $m = 479$. Now Col. Cunningham's Binary Canon gives, on page 50, a Table of Residues (R) for $p = 479$. From this we extract

x	0	1	2	3	4	5	6	7	8	9
.	1	2	4	8	16	32	64	128	256	33
1	66	132	264	49	98	196	392	305	131	262
2	45	90	180	360	241	3	6	12	24	48

hence

$$(2^{29} - 48) \div 479 = 10001000111010001100000 \text{ (in the binary scale)}$$
$$= 1120816 \text{ (in the denary scale).}$$

[Verification:—

$$2^{29} = 536870912 = 479 \times 1120816 + 48.]$$

It will be noticed that, starting from the power of 2 (viz. 2^9) just greater than the modulus ($2^9 > 479$), the quotient is obtained simply by writing unity for this residue (which is odd), and, following this in order to the right, unity or zero for each successive residue according as that residue is odd or even. The last residue used, being that of the power used as dividend (*e.g.* in the case considered the 29th), gives the Remainder R_n after the division into 2^{29}.

5. For other bases than 2 and 10 a slight modification of the procedure detailed above appears to lead most readily to the required result.

It is, of course, necessary to have a TABLE of residues of powers of that base for various moduli. The powers included should run to 30 at least, and the moduli (primes and powers of primes being the most generally useful) should run to < 1000.†

---

* Such a Table is given in *A Binary Canon showing residues of powers of* 2 *for divisors under* 1000, by Lt.-Col. A. Cunningham, R.E., London, 1900. It has now been extended (as yet only in MS.), so as to give R_x (mod. m), where $x > 70$ and m includes all primes and powers of primes below 10,000.

† Besides the Binary and Denary Canons already mentioned, the only Tables known to the Author have for their bases 3 and 5. These, compiled by himself (and at present only in MS.), show residues of 3^x and 5^x, up to $x > 30$ for all prime and powers of prime moduli up to 1409.

The simplest manner of computing, in a scale of given radix (a), the successive digits (q_n) of the quotient of a power (a^x) of that radix by a given divisor (m), by use of a Table of Residues (R_x) of a^x, appears to arise from consideration of the identity which serves most generally for the compilation of such Tables.

Here, R_{n-1} and R_n being respectively the Residues of the powers $n-1$ and n of a, we have

$$a^n \equiv aR_{n-1} \equiv R_n \pmod{m}.$$

Now suppose that

$$q_n m < aR_{n-1} < (q_n + 1)\, m, \quad (q_n \text{ integral}) \ldots\ldots (8),$$

then, dividing by a,

$$\frac{q_n m}{a} < R_{n-1} < \frac{(q_n + 1)\, m}{a} \quad \ldots\ldots\ldots\ldots (9).$$

From (8) we see that q_n is the *proper* digit (q_n) of the quotient (Q_n) corresponding to the remainder R_n.

From (9), we see that q_n is the integer next below $aR_{n-1} \div m$, and that it can be readily obtained direct from an auxiliary Table arranged thus

$q =$	0	1	2	..	q_n	$q_n + 1$	..	$a-1$	a
$qm \div a =$	0	α	β	..	κ	κ_1	..	$m-\alpha$	m

(the numbers in the second row being the values of $qm \div a$ calculated as decimal fractions to *one* place only).

Our auxiliary equation (9) now shows that if

$$\kappa < R_{n-1} < \kappa_1,$$

then q_n will be the *proper* digit of the quotient corresponding to the Remainder R_n. [In this form of the theorem it appears that the scrutiny of *one* remainder furnishes the digit of the quotient corresponding to the *next* remainder.]

Ex. Find the quotient when 3^{29} is divided by 479.

The residues of the first 29 powers of 3 for mod. 479 are as follows*:

x	0	1	2	3	4	5	6	7	8	9
	1	3	9	27	81	243	250	271	334	44
1	132	396	230	211	154	462	428	326	20	60
2	180	61	183	70	210	151	453	401	245	256

* Extracted from the Author's MS. Table above quoted.

Also

$q =$	1	2	3
$qm \div a =$	159.7	319.3	479

gives the scale for the quotient digits.

Hence

$(3^{39} - 256) \div 479 = 111200211022200101010221$ (in the ternary scale)
$= 143278449613$ in the denary scale.

[Verification:—

$3^{39} = 68630377364883 = 479 \times 143278449613 + 256$].

6. Conversely, given the quotient (Q_n) of a power (n) of base a, when divided by a modulus (m), we can find the corresponding residue.

We have $a^n = mQ_n + R_n$.....(1),

Now let a^k be the power of a next greater than m. We have

$$0 \equiv mQn + R_n, \quad \therefore R_n \equiv -mQ_n \pmod{a^k}.....(10),$$

whence R_n can be readily obtained. [The principal use of this proposition will be in the verification of the accuracy of residues already given in a Table.]

7. In the foregoing theory we have limited ourselves to the consideration of divisors such as occur (as moduli) in Tables of Residues. Our results can, however, be applied to lessen the work of division in cases where the divisor can be divided by one factor (prime or power of prime) not beyond the power of the table. The work in this case will be:—

First, resolve the divisor $d = ms$, where m is the largest factor of d occurring, in the Table, as modulus.

Second, determine $a^n = mQ_n' + R_n'$, then, successively, reduce Q_n' to the denary scale and determine

$$Q_n' = sQ_n + R_n''.$$

Finally,

$$a^n = msQ_n + (mR_n'' + R_n')$$
$$= msQ_n + R_n,$$

where $R_n = mR_n'' + R_n'$.

8. If the period (ξ) of the base (a) for the proposed modulus is less than the power (n) to be divided, the Table may not show, explicitly, a residue of this power. In such a case it must be remembered that

$$a^{\xi+1} \equiv a, \ a^{\xi+2} \equiv a^2, \ a^{\xi+3} \equiv a^3, \text{ etc. (mod. } m),$$

accordingly $a^n \equiv a^{n-\xi} \equiv a^{n-2\xi}$, etc. (mod. m).

Hence the Residues tabulated must be used to obtain the the successive digits of the quotient, then, after using $R_\xi (=1)$, follow on with R_1, R_2, &c., as far as $R_{n-\xi-1}$ (the residue of a^{n-1}), which, in the more general form of the theorem, furnishes the criterion for the determination of the final digit (q_n) of Q_n.

EXTENSION OF A FUNDAMENTAL THEOREM IN GROUP THEORY.

By *G. A. Miller.*

THE theorem, *every group of order p^m contains an Abelian subgroup of order p^α whenever* $m > \frac{\alpha(\alpha-1)}{2}$ was proved in this Journal, Vol. XXVII., 1898, p 119. It should then have been noted that the proof applies to the case when this Abelian subgroup is invariant. In view of the recent theorem* that every non-cyclic group of order $p^m (p>2)$ contains a non-cyclic invariant subgroup of order p^β, where β may have any value from 2 to m, it seems desirable to state the theorm mentioned above in a more general form, as follows:—

Every non-cyclic group of order p^m contains an invariant Abelian subgroup of order p^α, where α may have any value consistent with $m > \frac{\alpha(\alpha-1)}{2}$. *Such a subgroup is non-cyclic whenever $\alpha > 1$ and $p > 2$.*

The apparent restriction to non-cyclic groups is of no consequence, since all cyclic groups are Abelian. The former proof applies directly to the present form of this theorem if the theorem in reference to non-cyclic invariant subgroups is employed. It may be added that the theorem in question is not correctly stated by Easton in his *Constructive Development of Group Theory*, 1902, p. 62. It is hoped that the present note exhibits the matter with sufficient clearness to avoid the incorrect inferences to which the original article was open.

Bulletin of the American Mathematical Society, Vol. X., 1904, p. 497.

Now Ready. Price 12s. net.

TABLES OF QUADRATIC PARTITIONS

of all PRIMES up to 100,000.

By Lt.-Col. ALLAN CUNNINGHAM, R.E.

FRANCIS HODGSON, 89 Farringdon Street, London, E.C.

CONTENTS.

PAGE

Determination of successive high primes (continued). By Lt.-Col. Cunningham and H. J. Woodall - - - - - - - - 81

On synthetic division. By H. J. Woodall - - - - - - - 90

Extension of a fundamental theorem in group theory. By G. A. Miller - 96

Papers for insertion should be sent to Dr. Glaisher, Trinity College, Cambridge, or to Messrs. Metcalfe and Co. Limited, Printing Office, Trinity Street, Cambridge.

An index to vols. i—xxv has been published. It is issued gratis to subscribers, but may be obtained separately, price Two Shillings.

No. CCCCIII.] NEW SERIES. [November, 1904.

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XXXIV.—NO. 7.

MACMILLAN AND CO.
London and Cambridge.
1904.

Price—One Shilling.

METCALFE AND CO. LIMITED, CAMBRIDGE.

ON THE ZEROES OF A CLASS OF INTEGRAL FUNCTIONS.

By *G. H. Hardy*, Trinity College, Cambridge.

§ 1. In a paper recently published in the *Proc. Lond. Math. Soc.** I showed how, by a development of an idea originally due to M. Borel†, very precise information could be obtained concerning the zeroes of a class of functions of the type

$$\Sigma \frac{x^{\phi(x)}}{\{\phi(n)\}!}. \tag{1}$$

In the present note I propose to discuss another class of functions, the nature of whose zeroes may be determined by treatment of an equally simple character.

The typical case of a function of the type (1) was that in which $\phi(n) = n^3$. I found that the zeroes of this function lay (roughly speaking) on a series of concentric circles, the number of zeroes associated with each circle increasing indefinitely with the radius of the circle, and their distribution round the circle being like that of the angular points of a regular polygon. The distribution of the zeroes of the functions considered here is entirely different; they will be found, in the most typical case, to be associated not with a series of circles, but with one straight line, the negative real axis.

§ 2. Consider the function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots, \tag{2}$$

where

$$\frac{1}{a_\nu} = b_0 b_1 b_2 \ldots b_\nu, \tag{3}$$

b_ν being the value for $\xi = \nu$ of a function $b(\xi)$ which is positive and continuous for all positive values of ξ, and increases steadily to ∞ with ξ. Let

$$x = re^{i\theta}, \quad t_\nu = a_\nu x^\nu, \quad \tau_\nu = |t_\nu|, \tag{4}$$

* Vol. II., p. 332.

† *Acta Mathematica*, XX., p. 391.

and suppose that

$$b_n \leqq r < b_{n+1}, \tag{5}$$

n being a large positive integer. Then, since

$$\frac{\tau_{\nu+1}}{\tau_\nu} = \frac{r}{b_{\nu+1}}, \tag{6}$$

it is clear that the term t_n is numerically greater than any other save t_{n-1}, which may be equal to it.

Now

$$f(x) = t_n + t_{n+1}\left(1 + \frac{t_{n+2}}{t_{n+1}} + \frac{t_{n+3}}{t_{n+1}} + \ldots\right) + t_{n-1}\left(1 + \frac{t_{n-2}}{t_{n-1}} + \ldots + \frac{t_0}{t_{n-1}}\right). \tag{7}$$

But

$$\frac{\tau_{n+2}}{\tau_{n+1}} = \frac{r}{b_{n+2}} < \frac{b_{n+1}}{b_{n+2}},$$

$$\frac{\tau_{n+3}}{\tau_{n+1}} = \frac{r^2}{b_{n+2}b_{n+3}} < \left(\frac{b_{n+1}}{b_{n+2}}\right)^2,$$

and so on. Hence

$$\left|\frac{t_{n+2}}{t_{n+1}} + \frac{t_{n+3}}{t_{n+1}} + \ldots\right| < \frac{b_{n+1}}{b_{n+2} - b_{n+1}}. \tag{8}$$

Again

$$\frac{\tau_{n-2}}{\tau_{n-1}} = \frac{b_{n-1}}{r} \leqq \frac{b_{n-1}}{b_n},$$

$$\frac{\tau_{n-3}}{\tau_{n-1}} = \frac{b_{n-1}b_{n-2}}{r^2} < \left(\frac{b_{n-1}}{b_n}\right)^2,$$

and so on. Hence

$$\left|\frac{t_{n-2}}{t_{n-1}} + \frac{t_{n-3}}{t_{n-1}} + \ldots\right| < \frac{b_{n-1}}{b_n - b_{n-1}}. \tag{9}$$

From (7), (8), and (9) we deduce

$$f(x) = t_{n-1}(1+\theta) + t_n + t_{n+1}(1+\theta'), \tag{10}$$

where

$$|\theta| < \frac{b_{n-1}}{b_n - b_{n-1}}, \quad |\theta'| < \frac{b_{n+1}}{b_{n+2} - b_{n+1}}. \tag{11}$$

§ 3. Let us consider in particular the case in which

$$r = b\,(n+\tfrac{1}{2}).$$

Then

$$\frac{\tau_{n-1}}{\tau_n} = \frac{b_n}{b_{n+\frac{1}{2}}} \tag{12}$$

—writing $b_{n+\frac{1}{2}}$ for $b\,(n+\frac{1}{2})$—and

$$\frac{\tau_{n+1}}{\tau_n} = \frac{b_{n+\frac{1}{2}}}{b_{n+1}}. \tag{13}$$

Let us now introduce the further condition that

$$b\,(\xi+\tfrac{1}{2}) \geqq 3b\,(\xi). \tag{14}$$

Then

$$\frac{\tau_{n-1}}{\tau_n} \leqq \tfrac{1}{3}, \quad \frac{\tau_{n+1}}{\tau_n} \leqq \tfrac{1}{3}, \tag{15}$$

and, as $b_{n+2} \geqq 9b_{n+1}, \quad b_n \geqq 9b_{n-1}$,

$$\frac{b_{n-1}}{b_n - b_{n-1}} \leqq \tfrac{1}{8}, \quad \frac{b_{n+1}}{b_{n+2} - b_{n+1}} \leqq \tfrac{1}{8}, \tag{16}$$

and

$$f(x) = t_n(1+\phi), \tag{17}$$

where

$$|\phi| < 2 \cdot \tfrac{1}{3} \cdot \tfrac{2}{8} < 1.$$

It follows by a well-known theorem that $f(x)$ *has exactly* n *roots within the circle*

$$r = b_{n+\frac{1}{2}},$$

and therefore has exactly *one* root between two successive such circles.

Moreover, when $x = -b_{n+\frac{1}{2}}$, $f(x)$ has the sign of t_n, *i.e.* of $(-1)^n$, and when $x = -b_{n+\frac{3}{2}}$ it has the sign of $(-1)^{n+1}$. Hence $f(x)$ has a real negative zero lying between the circles $r = b_{n+\frac{1}{2}}$ and $r = b_{n+\frac{3}{2}}$; and as it has but one zero between these circles all its roots are real and negative. As it is of *genre* zero the same is true of all its derivatives, as we know from a theorem due to Laguerre.*

* *Œuvres de Laguerre*, t. I., p. 167.

If the conditions imposed on b_n are only satisfied after a certain n the above statements are still exactly true for all zeroes whose modulus is greater than a certain r. They are thus true of any function

$$f(x)+P(x),$$

where $P(x)$ is a polynomial.

It is easy to see that the necessary conditions*

$$(n+1)\,a_{n+1}a_{n-1}-na_n^2<0$$

are satisfied.

Suppose, *e.g.* $$a_n=q^{n^2},$$

$$b_n=\frac{a_{n-1}}{a_n}=q^{-2n+1},$$

$$\frac{b_{n+\frac{1}{2}}}{b_n}=\frac{1}{q}.$$

Thus, if $q\leqq\frac{1}{3}$, the function

$$\sum_0^\infty q^{\nu^2}x^\nu \tag{18}$$

has exactly n roots inside the circle $r=q^{-2n}$, and these roots are all real and negative.

§ 4. I return now to the equation (10), and make the further assumption that if ω is any fixed positive quantity, however small,

$$\lim_{\xi=\infty}\frac{b_{\xi+\omega}}{b_\xi}=\infty. \tag{19}$$

We may then replace (10) by

$$f(x)=t_{n-1}(1+\epsilon)+t_n+t_{n+1}(1+\epsilon'), \tag{10'}$$

where ϵ, ϵ' are very small when n is very large. I may remark that M. E. le Roy, considering only *real positive* values of x, has established a similar equation for a class of functions defined in a way somewhat different from mine. His analysis, however, is less simple and direct.†

* Laguerre, *loc. cit.*, p. 171; Borel, *Leçons*, II., p. 35.

† E. le Roy, "Valeurs asymptotiques de certaines séries procédant suivant les puissances entières et positives d'une variable réelle" (*Bull. des Se. Math.*, XXIV., p. 245).

Let δ be any fixed small positive quantity. Then we deduce immediately from (10′) that

$$(20) \qquad f(x) = t_n(1+\epsilon), \quad (b_{n+\delta} \leqq r \leqq b_{n+1-\delta}),$$

$$(20') \qquad f(x) = t_{n-1}(1+\epsilon) + t_n(1+\epsilon'), \quad (b_n \leqq r \leqq b_{n+\delta}),$$

$$(20'') \qquad f(x) = t_n(1+\epsilon) + t_{n+1}(1+\epsilon'), \quad (b_{n+1-\delta} \leqq r \leqq b_{n+1}).$$

It is clear that (20′) holds throughout the range of values

$$(21) \qquad b_{n-\delta} \leqq r \leqq b_{n+\delta},$$

and similarly for (20″).

§ 5. Now $f(x)$ has one, and only one, zero in the region defined by (21). At this zero

$$0 = t_{n-1}(1+\epsilon) + t_n(1+\epsilon'),$$

or

$$(22) \qquad x = -b_n(1+\epsilon).$$

It is easy to find an upper limit for the modulus of the real and small quantity ϵ; but the equation (22) gives us sufficient information for our present purpose.

Suppose, *e.g.*

$$a_n = e^{-n^3},$$

$$b_n = e^{3n^2-3n+1},$$

$$\frac{b(\xi+\omega)}{b(\xi)} = e^{6\omega\xi-3\omega+3\omega^2},$$

and the condition (19) is evidently satisfied. The same is true for $a_n = e^{-n^\alpha}$ $(\alpha > 2)$. Another example is given by

$$1/a_n = 1^1 2^2 3^3 \ldots n^n.$$

The last function is interesting; its zeroes are given by

$$x = -n^n(1+\epsilon),$$

its increase by

$$M(r) = r^{\frac{1}{2}\frac{\log r}{\log\log r}(1+\epsilon)}.$$

§ 6. In conclusion it may be pointed out that the hypothesis that a_n is real and positive is by no means essential. This assumption, however, leads to the simplest and most tangible results.

NOTE ON A FORMER PAPER.

By *G. H. Hardy*, Trinity College, Cambridge.

I FIND that the contents of §§ 2 and 3 of my paper "A generalisation of Frullani's Integral"* are practically identical with those of a paper written by Prof. Lerch.† The only difference is that (in my notation) Prof. Lerch supposes that

$$\lim_{u=\infty} \frac{\phi(u)}{u^{m-1}} = \alpha,$$

and obtains the result

$$\left\{\alpha - \frac{\phi^{(m-1)}(0)}{(m-1)!}\right\} \Sigma (A a^{m-1} \log a),$$

while I suppose that

$$\int^{\infty} \frac{\phi(u)}{u^m}\, du$$

is convergent, and obtain the result

$$-\frac{\phi^{(m-1)}(0)}{(m-1)!} \Sigma (A a^{m-1} \log a).$$

The two formulæ agree if (for example)

$$|\phi(u)| < K u^{m-1-\delta} \quad (\delta > 0),$$

when $\alpha = 0$, but each includes a certain number of cases excluded by the other. Thus mine would exclude all cases considered by Prof. Lerch in which $\alpha \neq 0$, while his would exclude all the ordinary cases in which the integral above written is conditionally convergent. The two results correspond in fact to the two well-known cases of Frullani's theorem, and must be taken together to give the complete generalisation.

---

* *Messenger*, Vol. XXXIV., p. 11.

† "Généralisation du théoreme de Frullani," *Sitzungsberichte d. k. Böhmischen Gesellshaft der Wiss.*, June 2, 1893.

ON CERTAIN DOUBLY INFINITE SYSTEMS OF TWISTED POLYGONS.

By *R. A. Roberts.*

I PURPOSE to consider here some properties of doubly infinite systems of polygons inscribed in a quadric. Let $p =$ a constant, $q =$ a constant denote the two systems of generators of a quadric, then it is evident that the coordinates of any point X_1, X_2, X_3, X_4 on the surface can be expressed thus

$$X_1 : X_2 : X_3 : X_4 = \phi_1 : \phi_2 : \phi_3 : \phi_4 \quad \ldots\ldots\ldots\ldots (1),$$

where $$\phi_r = a_r pq + b_r p + c_r q + d\delta.$$

Now it is evident that if we substitute for p, q homographic transformations thereof, viz. $\frac{\alpha p + \beta}{\gamma p + \delta}$, $\frac{\alpha' q + \beta'}{\gamma' q + \delta'}$, for p, q respectively in (1), we shall reproduce equations of precisely the same form. Hence we can transform all relations connecting p and q respectively in the same way as binary quantics. For instance, suppose two points p_1, q_1; p_2, q_2 on the surface are such that

$$Ap_1p_2 + B(p_1 + p_2) + C = 0, \quad A'q_1q_2 + B'(q_1 + q_2) + C' = 0 \ldots (2),$$

then by homographic transformations it is evident that these can be reduced to the forms

$$p_1 + p_2 = 0, \quad q_1 + q_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

But from (1) we shall evidently have

$$X_1 : X_2 : X_3 : X_4 = pq : p : q : 1 \ldots\ldots\ldots\ldots (4),$$

where $$X_1 = \begin{vmatrix} x_1, & x_2, & x_3, & x_4 \\ b_1, & b_2, & b_3, & b_4 \\ c_1, & c_2, & c_3, & c_4 \\ d_1, & d_2, & d_3, & d_4 \end{vmatrix}, \quad X_2 = \&c.$$

Now subject to the relations (2) it is evident that the line joining the two points intersects the lines X_1X_4 and X_2X_3.

Again, these two lines are mutually conjugate with regard to the surface, and one is completely determined when the other is given. Hence when two points on the surface are such that the equations (2) are satisfied, the line joining them intersects two fixed lines mutually conjugate with regard to the surface. One of the latter lines is not limited in any way, for a line involves four constants, and four constants are involved in (2).

I observe now that, if two points on the surface are connected by two relations, the line joining them belongs to a congruency. The line thus is a bitangent of a surface, or touches two surfaces, or intersects two curves, &c., the latter conditions being degraded forms of the first. Now let us consider the two relations

$$\left.\begin{aligned} Ap_1p_2 + Bp_1 + Cp_2 + D = 0 \\ A'q_1q_2 + B'q_1 + C'q_2 + D' = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (5),$$

which involve six independent constants. Now by homographic transformations these relations may be reduced to

$$p_1 + \lambda p_2 = 0, \quad q_1 + \mu q_2 = 0 \ldots\ldots\ldots\ldots\ldots (6).$$

But taking in (4),

$$X_1 = z, \ X_2 = x, \ X_3 = y, \ X_4 = 1 \ldots\ldots\ldots\ldots (7),$$

the quadric becomes

$$z = xy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

and the relations (6) become

$$x_1 + \lambda X_2 = 0, \quad y_1 + \mu Y_2 = 0 \ldots\ldots\ldots\ldots (9).$$

Now if the line joining two points touches the surface $\theta z = xy$, we have

$$\{\theta(z_1 + z_2) - x_1y_2 - y_1x_2\}^2 - 4(\theta z_1 - x_1y_1)(\theta z_2 - x_2y_2) = 0 \ldots (10);$$

and this, if the points lie on (8) and are subject to the conditions (9), reduces to

$$\{\theta(1 + \lambda\mu) + \lambda + \mu\}^2 - 4\lambda\mu(1 - \theta)^2 = 0 \ldots\ldots (11).$$

This gives two constant values of θ, so that subject to the conditions (9) the line touches the two quadrics

$$\theta_1 z - xy = 0, \quad \theta_2 z - xy = 0 \ldots\ldots\ldots\ldots (12),$$

where θ_1, θ_2 are the roots of the quadratic (11). These two quadrics evidently intersect the given quadric and each other in the sides of the quadrilateral $X_1X_4=0$, $X_2X_3=0$, and have the same tangent planes at the four vertices. Hence we see that, if two points on the surface are such that the two relations (5) are satisfied, the line joining them touches two quadrics which are both inscribed in the given surface at the same four points.

From these results I deduce the following theorem concerning polygons inscribed in the surface. If a polygon of n sides be inscribed in the surface so that $n-1$ sides intersect, respectively, $n-1$ pairs of lines mutually conjugate with regard to the surface, then the n^{th} side touches two quadrics touching the given surface at the same four points.

For if we have $n-1$ pairs of relations of the form (2) connecting p_1, p_{12}, p_2, &c., q_1, q_{12}, q_2, &c., then eliminating p_{12}, p_2, ..., p_{n-1}, q_{12}, q_2, ..., q_{n-1}, we get two relations of the form (5) connecting p_1, p_n and q_1, q_n respectively, and this proves what I have just stated.

If two relations are satisfied, the two quadrics reduce to a pair of mutually conjugate lines. Hence we can have a doubly infinite system of polygons of n sides inscribed in the surface, so that the sides touch n pairs of mutually conjugate lines, respectively, provided these latter lines are connected by six relations. Again, if a polygon of n sides be inscribed in the surface so that $n-1$ sides touch pairs of quadrics touching the surface in four points, the points of contact of each member of a pair being the same, then the n^{th} side touches another pair of quadrics related in the same way to the surface. This follows from the equations (5) in the same way as in the previous case.

It may be observed that in the preceding cases the diagonals, or lines joining non-consecutive vertices of the polygons, touch, in general, pairs of quadrics inscribed in the surface at four points, for in each case we obtain by elimination pairs of relations of the form (5) connecting the coordinates of such vertices.

Suppose we want to find inscribed polygons whose sides all touch the same pair of inscribed quadrics, we may most conveniently write the given quadric $x^2+y^2=1+z^2$, and the touching quadrics $x^2+y^2=k^2(1+z^2)$, $x^2+y^2=k'^2(1+z^2)$. We find then $k=\dfrac{\cos\alpha}{\cos\beta}$, $k'=\dfrac{\sin\alpha}{\sin\beta}$, where α, β are different values of $\dfrac{r\pi}{n}$, where n is the number of sides of the polygon

and r is an integer. Hence there is no doubly infinite system of polygons with less than five sides.

I now proceed to consider pairs of points on the surface $p_1, q_1; p_2, q_2$, such that

$$\left.\begin{aligned} Ap_1q_2 + Bp_1 + Cq_2 + D &= 0 \\ A'p_2q_1 + B'p_2 + C'q_1 + D' &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots(13).$$

Considering the pairs of generators at the points p_1, q_1; p_2, q_2, it is evident that they intersect in the points p_2, q_1; p_1, q_2.

Hence from (13) we see that the latter points satisfy relations of the form

$$Apq + Bp + Cq + D = 0 \ldots\ldots\ldots\ldots(14).$$

But this, it is easy to see, expresses that these points lie on fixed planes. Hence the plane containing intersecting generators at p_1, q_1; p_2, q_2 touches the surface along a plane section, and therefore is a tangent plane of the cone circumscribed to the surface along that section. Hence the line joining p_1, q_1; p_2, q_2, as it lies in that plane, touches the circumscribed cone. Thus the equations (13) express that the line joining p_1, q_1 touches two circumscribed cones. These cones are arbitrary, for a circumscribed cone involves three constants and six constants are involved in (13). Considering now polygons of n sides, we must separate the cases in which n is odd and n is even. If n is odd and $n-1$ sides touch $n-1$ pairs of circumscribed cones, respectively, the n^{th} side touches a pair of quadrics touching the given surface at four points. Again, if n is even and $n-1$ sides touch $n-1$ pairs of circumscribed cones, respectively, the n^{th} side touches another pair of circumscribed cones. In both these cases the lines joining alternate vertices of the polygons touch pairs of quadrics inscribed in the given surface at the same four points. These results follow easily by elimination from the equations such as (13), which are written down for the $n-1$ sides of the polygon.

It is to be observed that, as a particular case, if the coefficients in each member of the equations (13) are the same, the cones coincide and the line joining p_1, q_1; p_2, q_2 simply passes through a fixed point. Hence, if $n-1$ sides of an inscribed polygon of n sides pass through $n-1$ fixed points, respectively, the n^{th} side touches a pair of inscribed quadrics or touches a pair of circumscribed cones, according

as n is odd or even. We see thus that the problem to describe an inscribed polygon so that the n sides pass, respectively, through n fixed points admits of four solutions, for we can draw through a point four common tangent lines to two quadrics or two cones.

In the case of the triangle it is to be observed that one of the quadrics degenerates to a line, viz. one of the diagonals of the quadrilateral common to the other quadric and the given surface, so that the third side intersects this fixed line and touches an inscribed quadric. As a particular case I may notice the known result that a doubly infinite number of quadrilaterals can be inscribed in the surface, so that each side passes through a vertex of a given self-conjugate tetrahedron.

I now consider the more general relations

$$(a, b, c, f, g, h)(p_1p_2, p_1+p_2, 1)^2 = 0, \quad Aq_1q_2 + B(q_1+q_2) + C = 0 \quad \text{.....(15).}$$

The latter of these relations can by homographic transformation be reduced to the form $q_1 + q_2 = 0$. Taking then the surface in the form $z = xy$, where $x = p$, $y = q$. I propose to find the surface touched by the line $p_1, q_1; p_2, q_2$, subject to the conditions (15).

Let $x_1 + x_2 = 2\alpha$, $x_1 - x_2 = 2\beta$, then for any point on the line joining $x_1, y_1; x_2, y_2$, we have, observing that $y_1 + y_2 = 0$,

$$x = \alpha + \theta\beta, \quad y = \theta y_1, \quad z = y_1(\beta + \theta\alpha),$$

whence
$$x = \alpha + \theta\beta, \quad \frac{z}{y} = \alpha + \frac{\beta}{\theta} \text{ .....(16).}$$

To find the points of contact with the surface I differentiate these equations regarding x, y, z as constants. Eliminating then $d\theta$, I obtain

$$\theta^2 + z\theta\frac{d\beta}{d\alpha} + 1 = 0 \text{ .....(17),}$$

Now $\alpha^2 - \beta^2 = x_1x_2 = \gamma$, so that from (16) we get

$$x + \frac{z}{y} = \frac{d\gamma}{d\alpha}, \quad \frac{xz}{y} = \frac{\alpha d\gamma}{d\alpha} - \gamma \text{ .....(18).}$$

But from (15) we may take

$$\alpha : \gamma : 1 = f_1 : f_2 : f_3,$$

where f_1, f_2, f_3 are quadratic expressions in a parameter λ, so that we obtain

$$x + \frac{z}{y} = \frac{J_{12}}{J_{23}}, \quad \frac{xz}{y} = \frac{J_{13}}{J_{23}} \quad \ldots\ldots\ldots\ldots\ldots(19),$$

where J_{pq} is the Jacobian of f_p and f_q. Hence we obtain a relation of the second degree connecting $x + \frac{z}{y}$, $\frac{xz}{y}$, and 1.

This represents a Kummer's quartic, viz. a quartic surface with two non-intersecting double lines, which are in this case u, x; z, y, that is, two generators of the given quadric. It is a ruled surface generated by lines intersecting the two double lines.

We can now find a doubly infinite number of polygons, of n sides, inscribed in the surface so that each side is a bitangent of a Kummer's quartic with two generators of the quadric as double lines. If $n-1$ sides satisfy relations of the form $A + B(q_1 + q_2) + Cq_1q_2 = 0$, then subject to a single condition the n^{th} side can satisfy an equation of the same form. Again, the equation

$$(a, b, c, f, g, h)\,(p_1p_2,\ p_1 + p_2,\ 1)^2 = 0 \ldots\ldots\ldots(20)$$

is equivalent to $u_1 - u_2 = \sigma$, where $u_2 = \int \frac{dp_r}{\sqrt{(P_r)}}$, an elliptic integral, and σ is a constant as is well known. For a closed polygon of n sides, we have

$$n\sigma = \omega \ldots\ldots\ldots\ldots\ldots\ldots\ldots(21),$$

where $\omega = 2\mu K + 2\gamma i k'$, $p = \dfrac{\alpha + \beta\,\mathrm{sn}^2 u}{\gamma + \delta\,\mathrm{sn}^2 u}$.

In this case the relation (20) is the same for all the pairs of variables p_r, p_{r+1}, but these may be different relations, provided P_r or the discriminant of (20) with regard to one of the variables is the same for all.

In the latter case there must be the further relation corresponding to

$$\sigma_1 + \sigma_2 + \ldots + \sigma_n = \omega \ldots\ldots\ldots\ldots(22).$$

I now consider a relation of the form

$$p_1^2(a + bq_2 + cq_2^2) + p_1(a' + b'q_2 + c'q_2^2) + a'' + b''q_2 + c''q_2^2 = 0$$
$$\ldots\ldots\ldots(23).$$

This evidently expresses that the point p_1, q_2 lies on the intersection of the given surface with another quadric, viz. a quadri-quadric curve. Now the tangent plane at this point contains the points $p_1, q_1; p_2, q_2$. The latter points, therefore, lie on a tangent plane whose point of contact is on a quadri-quadric curve. Such a tangent plane touches another quadric, that is, it is a tangent plane of a quadri-quadric developable circumscribed about the given surface. Hence the line joining $p_1, q_1; p_2, q_2$, subject to the condition (23), touches a circumscribed quadri-quadric developable.

Now it is well known that (23) leads to the equation

$$\frac{dp_1}{\sqrt{(P_1)}} + \frac{dq_2}{\sqrt{(Q_2)}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (24),$$

where P, Q have the same anharmonic function, which, if $\int \frac{dp}{\sqrt{P}}$, $\int \frac{dq}{\sqrt{Q}}$ are made proportional to u, v, respectively, u, v being elliptic integrals, gives

$$u_1 - v_2 = \sigma \quad \ldots\ldots\ldots\ldots\ldots\ldots (25).$$

Again, if the relation (23) holds also for p_2, q_1, then the line $p_1, q_1; p_2, q_2$ lies in another tangent plane of the same circumscribed quadri-quadric developable. We see now that we can have a doubly infinite number of polygons with an *even* number of sides inscribed in the surface, so that each side is a bitangent of a circumscribed quadri-quadric developable. For, from the equations

$$u_1 - v_2 = \sigma, \quad v_2 - u_3 = \sigma, \text{ \&c.} \ldots\ldots\ldots\ldots (26),$$

we get

$$n\sigma = \omega \quad \ldots\ldots\ldots\ldots\ldots\ldots (27);$$

and this condition also results from

$$u_2 - v_1 = \sigma, \quad v_3 - u_2 = \sigma, \text{ \&c.}$$

The developable thus has to satisfy one relation with the given surface. Again the sides may be bitangents of different developables, provided these surfaces all touch the same eight generators given by $PQ = 0$, that is, the two discriminants of (23) with regard to p_1 and p_2, respectively. These different developables then must be such that

$$\sigma_1 + \sigma_2 + \ldots + \sigma_n = \omega \quad \ldots\ldots\ldots\ldots\ldots (28).$$

Further, the sides of the polygon may all touch two different circumscribed quadri-quadric developables. Let u, v be the elliptic integrals corresponding to the relation (23), and let ϕ, ψ be the integrals corresponding to a relation of the same form with different coefficients, then from $u_1 - v_2 = \sigma$, &c., we get $n\sigma = \omega$, and from $\phi_1 - \psi_2 = \sigma'$, &c., we get

$$n\sigma' = \omega' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29).$$

Moreover, we can substitute for these two developables sets of developables of two different systems, each touching the same eight generators, respectively, and then we must have

$$\sigma_1 + \sigma_2 + \sigma_n = \omega, \ \sigma_1' + \sigma_2' + \sigma_n' = \omega' \quad \ldots\ldots\ldots (30).$$

In the case (26) for the quadrilateral it may be noticed that we have $u_1 - u_2 = \frac{1}{2}\omega$, $v_1 - v_2 = \frac{1}{2}\omega$, which shows that the diagonals intersect a pair of lines mutually conjugate with regard to the surface; for, by the theory of elliptic integrals, these relations are equivalent to

$$Ap_1p_2 + B(p_1 + p_2) + C = 0, \quad Aq_1q_2 + B(q_1 + q_2) + C = 0.$$

It may be observed that in the preceding cases the quadri-quadric curve may reduce to a twisted cubic; and then the circumscribed developable becomes a developable of the fourth degree, so that a quadri-quadric developable may be replaced by a developable of the fourth degree generated by the tangents of a twisted cubic. Again, a developable may be replaced by a circumscribed cone, and in the case in which the chord of the given surface is a bitangent of the developable, the latter surface may be replaced by a point.

I now proceed to consider the polygons from another point of view. Suppose a quadric to be written in the form

$$PQ - RS = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (31),$$

where P, Q, R, S are planes. Suppose this quadric to meet another quadric,

$$z - xy = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (32),$$

say, which touches these planes. Then, since each of the planes must be of the form $z + ay + bx + ab$, which becomes at the intersection $(x + a)(y + b)$, we see that for the curve of intersection there is a relation of the form

$$\phi(x)\psi(y) + C = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (33).$$

Now this can be written

$$(\phi+\lambda)\left(\psi+\frac{C}{\lambda}\right)+(\phi-\lambda)\left(\psi-\frac{C}{\lambda}\right)=0 \ldots\ldots(34),$$

where λ is arbitrary, and this shows that there is a singly infinite number of quadrilaterals inscribed in (31), so that the sides are generators of the quadric (32). Now let us consider a developable whose tangential equations are $\Sigma=0$, $\Sigma'=0$, touching the four planes P, Q, R, S. Then $\Sigma+\lambda\Sigma'=0$ also touches these four planes; but the generators of all the surfaces of the system $\Sigma+\lambda\Sigma'=0$ are bitangents of the developable $\Sigma=\Sigma'=0$. Hence we have a doubly infinite system of quadrilaterals inscribed in the quadric $PQ-RS=0$, so that each side is a bitangent of a quadri-quadric developable touching the planes P, Q, R, S. This developable may reduce to two cones of the second order intersecting on two planes. Again, let us consider the cubic

$$PQR-STU=0 \ldots\ldots\ldots\ldots\ldots(35),$$

where P, Q, R, S, T, U are planes, to which form the general cubic surface can be reduced in a finite number of ways. In precisely the same way we find that a doubly infinite number of hexagons can be inscribed in the cubic, so that each side is a bitangent of a quadri-quadric developable touching the six planes P, Q, R, S, T, U. This developable can reduce to two cones of the second order. Thus we can have one cone touching P, Q, R and another cone touching S, T, U, provided these two cones have two common tangent planes. The cubic, if P, Q, R are of the form $u-\alpha$, $u-\beta$, $u-\gamma$ respectively, and S, T, U are of the form $v-\alpha$, $v-\beta$, $v-\gamma$, becomes divisible by $u-v$, so that we have hexagons inscribed in a quadric so that the sides are bitangents of a quadri-quadric developable. Again, we have similar results for the quartic

$$PQRS-TUVW=0 \ldots\ldots\ldots\ldots(36),$$

and this, if the planes are of the form $u-\alpha$, $u-\beta$, $u-\gamma$, $u-\delta$, and $v-\alpha$, $v-\beta$, $v-\gamma$, $v-\delta$ respectively, becomes divisible by u, v, so that we have a doubly infinite system of octagons inscribed in a cubic surface, so that the sides are bitangents of a quadri-quadric developable touching the eight planes in (36).

In the preceding cases the quadrics may have a line in common, and then the developable reduces to a line and a developable of the fourth order generated by the tangents of a twisted cubic. The sides of the polygon then are either

bitangents of the developable, or intersect a fixed line and touch the developable once. Such a developable, it may be observed, can be described to touch six arbitrary planes.

I now proceed to show that a doubly infinite number of polygons can be inscribed in the surface

$$x^l y^m z^n = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (37),$$

so that each side touches two surfaces of the system

$$x^l y^m z^n = k \quad \ldots\ldots\ldots\ldots\ldots\ldots (38).$$

Let us take two points on (37), so that

$$x_2 = ax_1,\ y_2 = by_1,\ z_2 = cz_1 \quad \ldots\ldots\ldots\ldots (39),$$

where a, b, c are constants such that $a^l b^m c^n = 1$. Then, if x, y, z is any point on the line joining these two points, we have

$$x = \{1+(a-1)\,\theta\}\,x_1,\ y = \{1+(b-1)\,\theta\}\,y_1,\ z = \{1+(c-1)\,\theta\}\,x_1$$
$$\ldots\ldots\ldots\ldots (40).$$

Hence, substituting these values in (38), we get

$$\{1+(a-1)\,\theta\}^l\,\{1+(b-1)\,\theta\}^m\,\{1+(c-1)\,\theta\}^n = k \ldots (41).$$

Taking the discriminant of this with regard to θ, we determine the values of k giving the surfaces of the system (38) which touch the line joining x_1, y_1, z_1 and x_2, y_2, z_2. Now differentiating (41) with regard to θ, we get

$$\frac{l\,(a-1)}{1+(a-1)\,\theta} + \frac{m\,(b-1)}{1+(b-1)\,\theta} + \frac{n\,(c-1)}{1+(c-1)\,\theta} = 0 \ldots (42);$$

and this gives two values of θ, so that from (41) we have two values of k, and therefore two given surfaces of the system (38) touching the line joining x_1, y_1, z_1; x_2, y_2, z_2.

Thus, from (39), we see that we can have a doubly infinite number of polygons inscribed in (37), so that each side touches a pair of surfaces of the system (38), provided

$$a_1 a_2 a_3 - a_n = 1,\ b_1 b_2 - b_n = 1,\ c_1 c_2 - c_n = 1 \ldots . (43),$$

the third relation being contained in the first two. It may be observed that if the discriminant of the quadratic (42) in θ vanishes, the two corresponding surfaces (38) coincide, and the side of the polygon is then an inflexional tangent of a surface of the system (38).

CONTENTS.

	PAGE
On the zeroes of a class of integral functions. By G. H. HARDY - -	97
Note on a former paper. By G. H. HARDY - - - - - - -	102
On certain doubly infinite systems of twisted polygons. By R. A. ROBERTS	103

Papers for insertion should be sent to Dr. Glaisher, Trinity College, Cambridge, or to Messrs. Metcalfe and Co. Limited, Printing Office, Trinity Street, Cambridge.

An index to vols. i—xxv has been published. It is issued gratis to subscribers, but may be obtained separately, price Two Shillings.

No. CCCCIV.] NEW SERIES. [December, 1904

Stanford University Library

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XXXIV.—NO. 8.

MACMILLAN AND CO.
London and Cambridge.
1904.

Price—One Shilling.

METCALFE AND CO. LIMITED, CAMBRIDGE.

ON THE REDUCTION OF THE PROBLEM OF n BODIES.

By *T. L. Bennett.*

In Section 1 the order is reduced by 6 by means of the centre of gravity integrals. The method of this Section is a generalisation of that which M. Poincaré used for the problem of three bodies. This generalisation was actually made by M. Poincaré, but is here given for the sake of completeness.

In Section 2 the order is further reduced by 4 by means of the angular momentum integrals and the elimination of the node. The method is a generalisation of that used by Mr. Whittaker for the problem of three bodies.

In Section 3 it is pointed out that the integral of energy is soluble in one of the momenta, so that the usual method of reducing the order by 2 by means of the energy integral and the elimination of the time could be carried out. It has not, however, appeared to be worth while to give the actual reduction.

Notation. For the sake of conciseness, the mass of the ith particle has been denoted indifferently by m_{3i-2}, m_{3i-1}, or m_{3i}. The notation $\sum_{i,j=1}^{r}$ has been used to denote that the summation is to extend over all pairs i, j of unequal integers not greater than r.

§ 1. Reduction from order $6n$ to order $6n-6$ by means of the six centre of gravity integrals.

Let q_{3i-2}, q_{3i-1}, q_{3i} be coordinates of the ith particle referred to axes Ox, Oy, Oz fixed in space, and let p_{3i-2}, p_{3i-1}, p_{3i} be its corresponding momenta ($i = 1, 2, 3, \ldots, n$).

The equations of motion are

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \quad (r = 1, 2, \ldots, 3n) \quad \ldots\ldots(1),$$

where

$$H = \sum_{r=1}^{3n} \frac{p_r^2}{2m_r} - \sum_{i,j=1}^{n} \frac{m_{3i}m_{3j}}{r_{ij}} \quad \ldots\ldots\ldots\ldots(2),$$

where r_{ij} is distance between ith and jth particles expressed in terms of the q's.

Now let us make the contact transformation

$$q_r = \frac{\partial W}{\partial p_r}, \quad p_r' = \frac{\partial W}{\partial q_r'} \quad (r = 1, 2, \ldots, 3n) \ldots\ldots(3),$$

where

$$W = \left(\sum_{r=1}^{3n-3} q_r' p_r\right) + q'_{3n-2} \sum_{i=1}^{n} p_{3i-2} + q'_{3n-1} \sum_{i=1}^{n} p_{3i-1} + q'_{3n} \sum_{i=1}^{n} p_{3i} \ldots\ldots\ldots(4).$$

The equations (3) being formed, it becomes clear that

$$q'_{3n-2},\ q'_{3n-1},\ q'_{3n}$$

are coordinates of the nth particle referred to Ox, Oy, Oz;

$$q'_{3i-2},\ q'_{3i-1},\ q'_{3i}$$

are coordinates of the ith particle referred to parallel axes through the nth particle ($i = 1, 2, \ldots, n-1$);

$$p'_{3n-2},\ p'_{3n-1},\ p'_{3n}$$

are the components of the total momentum of the system;

$$p'_{3i-2},\ p'_{3i-1},\ p'_{3i}$$

are the components of the momentum of the ith particle ($i = 1, 2, \ldots, n-1$).

Thus the equations (1) become

$$\frac{dq_r'}{dt} = \frac{\partial H}{\partial p_r'}, \quad \frac{dp_r'}{dt} = -\frac{\partial H}{\partial q_r'} \quad (r = 1, 2, \ldots, 3n),$$

where

$$H = \sum_{r=1}^{3n-3} \frac{p_r'^2}{2m_r} + \frac{1}{2m_{3n}}\left\{\left(p'_{3n-2} - \sum_{i=1}^{n-1} p'_{3i-2}\right)^2 + \left(p'_{3n-1} - \sum_{i=1}^{n-1} p'_{3i-1}\right)^2 + \left(p'_{3n} - \sum_{i=1}^{n-1} p'_{3i}\right)^2\right\} - \sum_{i,j=1}^{n} \frac{m_{3i} m_{3j}}{r_{ij}},$$

r_{ij} being now expressed in terms of q_1', q_2', ..., q'_{3n-3}.

Since this form of H does not contain q'_{3n-2}, q'_{3n-1}, or q'_{3n}, we have

$$p'_{3n-2} = \text{const.}, \quad p'_{3n-1} = \text{const.}, \quad p'_{3n} = \text{const.}$$

Moreover, without loss of generality, we may consider only motion relative to the centre of gravity, *i.e.* we may take

$$p'_{3n-3}=0, \quad p'_{3n-2}=0, \quad p'_{3n}=0.$$

The equations thus become (the accents being suppressed)

$$\frac{dq_r}{dt}=\frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{dt}=-\frac{\partial H}{\partial q_r} \quad (r=1, 2, \ldots, 3n-3) \ldots (5),$$

where

$$H=\sum_{r=1}^{3n-3}\frac{p_r^2}{2m_r}+\frac{1}{2m_n}\{(\sum_{i=1}^{n-1}p_{3i-2})^2+(\sum_{i=1}^{n-1}p_{3i-1})^2+(\sum_{i=1}^{n-1}p_{3i})^2\}$$
$$-\sum_{i,j=1}^{n}\frac{m_{3i}m_{3j}}{r_{ij}} \ldots (6).$$

§ 2. Reduction of the system (5) of the $(6n-6)$th order to a system of the $(6n-10)$th order by means of the three integrals of angular momentum and the elimination of the node.

Let us make the contact transformation

$$q_r=\frac{\partial W}{\partial p_r}, \quad p_r'=\frac{\partial W}{\partial q_r'}, \quad (r=1, 2, \ldots, 3n-3),$$

where

$$W=\sum_{i=1}^{n-3}\{p_{3i-2}(q'_{3i-2}\cos q'_{3n-4}-q'_{3i-1}\cos q'_{3n-3}\sin q'_{3n-4}$$
$$+q'_{3i}\sin q'_{3n-3}\sin q'_{3n-4})$$
$$+p_{3i-1}(q'_{3i-2}\sin q'_{3n-4}+q'_{3i-1}\cos q'_{3n-3}\cos q'_{3n-4}-q'_{3i}\sin q'_{3n-3}\cos q'_{3n-4})$$
$$+p_{3i}\,(\quad 0\quad +q'_{3i-1}\sin q'_{3n-3}\quad +q'_{3i}\cos q'_{3n-3})\}$$
$$+p_{3n-3}(q'_{3n-3}\cos q'_{3n-4}-q'_{3n-2}\cos q'_{3n-3}\sin q'_{3n-4})$$
$$+p_{3n-2}(q'_{3n-3}\sin q'_{3n-4}+q'_{3n-2}\cos q'_{3n-3}\cos q'_{3n-4})$$
$$+p_{3n-3}(\quad 0\quad +q'_{3n-2}\sin q'_{3n-3}\quad)$$
$$+p_{3n-1}(q'_{3n-3}\cos q'_{3n-4}-q'_{3n-1}\cos q'_{3n-3}\sin q'_{3n-4})$$
$$+p'_{3n-4}(q'_{3n-3}\sin q'_{3n-4}+q'_{3n-1}\cos q'_{3n-3}\sin q'_{3n-4})$$
$$+p'_{3n-3}(\quad 0\quad +q'_{3n-1}\sin q'_{3n-3}\quad) \ldots\ldots\ldots\ldots\ldots (7).$$

Consider the system of axes Ox', Oy', Oz' with origin O at the nth particle, and direction cosines given by the scheme

	x	y	z
x'	$l_1 = \cos q'_{3n-4}$	$m_1 = \sin q'_{3n-4}$	$n_1 = 0$
y'	$l_2 = -\sin q'_{3n-4} \cos q'_{3n-3}$	$m_2 = \cos q'_{3n-4} \cos q'_{3n-3}$	$n_2 = \sin q_{3n-3}$
z'	$l_3 = \sin q'_{3n-4} \sin q'_{3n-3}$	$m_3 = -\cos q'_{3n-4} \sin q'_{3n-3}$	$n_3 = \cos q_{3n-3}$

The equations $q_r = \dfrac{\partial W}{\partial p_r}$ $(r = 1, 2, \ldots, 3n-3)$ show that referred to Ox', Oy', Oz',

$$q'_{3i-2},\ q'_{3i-1},\ q'_{3i}$$

are the coordinates of the ith particle $(i = 1, 2, \ldots, n-3)$,

$$q'_{3n-3},\ q'_{3n-2},\ 0$$

are the coordinates of the $(n-2)$th particle,

$$q'_{3n-3},\ q'_{3n-5},\ 0$$

are the coordinates of the $(n-1)$th particle.

The equations $p_r' = \dfrac{\partial W}{\partial q_r'}$ $(r = 1, 2, \ldots, 3n-5)$ show that

$$p'_{3i-2},\ p'_{3i-1},\ p'_{3i}$$

are the components of momentum of the ith particle in directions Ox', Oy', Oz' $(i = 1, 2, \ldots, n-3)$,

$$p'_{3n-3},\ p'_{3n-2}$$

are the components for the $(n-2)$th particle in directions Ox', Oy',

$$p'_{3n-3},\ p'_{3n-5}$$

are the components for the $(n-1)$th particle in directions Ox', Oy'.

Also we have

$$p'_{3n-4} = \sum_{i=1}^{n-3} (-p_{3i-2} q_{3i-1} + p_{3i-1} q_{3i-2}) + (-p_{3n-3} q_{3n-2} + p_{3n-2} q_{3n-3}) + (-p_{3n-3} q_{3n-5} + p_{3n-5} q_{3n-3}),$$

therefore

$-p'_{2n-4}$ = angular momentum of system about Oz.

And

$$p'_{2n-3}=\sum_{i=1}^{n-3} q'_{2i-1}(l_3p_{2i-2}+m_3p_{2i-1}+n_3p_{2i})-q'_{2i}(m_3p_{2i-2}+m_2p_{2i-1}+m_1p_{2i})$$
$$+q'_{2n-7}(l_3p_{2n-6}+m_3p_{2n-7}+n_3p_{2n-6})-0$$
$$+q'_{2n-5}(l_3p_{2n-5}+m_3p_{2n-4}+n_3p_{2n-3})-0,$$

therefore p'_{2n-3} = angular momentum about Ox'.

To transform H to the new variables, it will be necessary to find the momenta of the $(n-2)$th and $(n-1)$th particles in direction Oz' in terms of the accented variables. Let these momenta be γ, γ' respectively.

Suppressing the accents, we have

$$p_{2n-3}=\sum_{i=1}^{n-3}\begin{vmatrix} q_{2i-1}, & q_{2i} \\ p_{2i-1}, & p_{2i}\end{vmatrix}+\begin{vmatrix} q_{2n-7}, & 0 \\ p_{2n-7}, & \gamma\end{vmatrix}+\begin{vmatrix} q_{2n-5}, & 0 \\ p_{2n-5}, & \gamma'\end{vmatrix},$$

And

$-p_{2n-4}=n_3$ (angular momentum about Oz')

$+n_2$ (angular momentum about Oy')

$$=n_3\left\{\sum_{i=1}^{n-3}\begin{vmatrix} q_{2i-2}, & q_{2i-1} \\ p_{2i-2}, & p_{2i-1}\end{vmatrix}+\begin{vmatrix} q_{2n-8}, & q_{2n-7} \\ p_{2n-8}, & p_{2n-7}\end{vmatrix}+\begin{vmatrix} q_{2n-6}, & q_{2n-5} \\ p_{2n-6}, & p_{2n-5}\end{vmatrix}\right\}$$
$$+n_2\left\{\sum_{i=1}^{n-3}\begin{vmatrix} q_{2i}, & q_{2i-2} \\ p_{2i}, & p_{2i-2}\end{vmatrix}+\begin{vmatrix} 0, & q_{2n-8} \\ \gamma, & p_{2n-8}\end{vmatrix}+\begin{vmatrix} 0, & q_{2n-6} \\ \gamma', & p_{2n-6}\end{vmatrix}\right.$$

Thus γ, γ' are given by the two linear equations

$$\left.\begin{aligned}\gamma\cdot q_{2n-7}+\gamma'\cdot q_{2n-5}&=p_{2n-3}-\sum_{i=1}^{n-3}\begin{vmatrix} q_{2i-1}, & q_{2i} \\ p_{2i-1}, & p_{2i}\end{vmatrix},\\ \gamma\cdot q_{2n-8}+\gamma'\cdot q_{2n-6}&=p_{2n-4}\operatorname{cosec} q_{2n-3}+\sum_{i=1}^{n-3}\begin{vmatrix} q_{2i}, & q_{2i-2} \\ p_{2i}, & p_{2i-2}\end{vmatrix}\\ &\quad+\cot q_{2n-3}\left\{\sum_{i=1}^{n-3}\begin{vmatrix} q_{2i-2}, & q_{2i-1} \\ p_{2i-2}, & p_{2i-1}\end{vmatrix}+\begin{vmatrix} q_{2n-8}, & q_{2n-7} \\ p_{2n-8}, & p_{2n-7}\end{vmatrix}+\begin{vmatrix} q_{2n-6}, & q_{2n-5} \\ p_{2n-6}, & p_{2n-5}\end{vmatrix}\right\}\end{aligned}\right\}$$
$$\ldots\ldots\ldots(8).$$

It must be noticed that these equations (8) do not involve q_{2n-4}.

Having regard to the physical meaning of the terms in the expression (6) for H, we can now express H in terms of our new variables.

$$H=\sum_{i=1}^{3n-9}\frac{p_r^2}{2m_r}+\frac{1}{2m_{3n-6}}(p^2_{3n-8}+p^2_{3n-7}+\gamma^2)+\frac{1}{2m_{3n-3}}(p^2_{3n-6}+p^2_{3n-5}+\gamma'^2)$$
$$+\frac{1}{2m_{3n}}\{(\sum_{i=1}^{n-3}p_{3i-2}+p_{3n-8}+p_{3n-6})^2+(\sum_{i=1}^{n-3}p_{3i-1}+p_{3n-7}+p_{3n-5})^2$$
$$+(\sum_{i=1}^{n-1}p_{3i}+\gamma+\gamma')^2\}-\sum_{i,j=1}^{n}\frac{m_{3i}m_{3j}}{r_{ij}}\quad\ldots(9).$$

The expressions r_{ij} being now expressed in terms of $q_1, q_2, \ldots, q_{3n-5}$.

The equations (5) become

$$\frac{dq_r}{dt}=\frac{\partial H}{\partial p_r},\quad \frac{dp_r}{dt}=-\frac{\partial H}{\partial q_r}\ (r=1, 2, \ldots, 3n-3)\quad\ldots(10),$$

where H stands for the expression (9).

The expression (9) for H does not involve q_{3n-4}, therefore the equations (10) have an integral

$$p_{3n-4}=k,$$

and q_{3n-4} only occurs in the equation

$$\frac{dq_{3n-4}}{dt}=\frac{\partial H}{\partial p_{3n-4}}.$$

Therefore q_{3n-4} drops out of the problem for the present, and may finally be obtained by a quadrature.

Thus we have only to solve the system of the $(6n-8)$th order

$$\frac{dq_r}{dt}=\frac{\partial H}{\partial p_r},\quad \frac{dp_r}{dt}=-\frac{\partial H}{\partial q_r}\ (r=1, 2, \ldots, 3n-5, 3n-3),$$

where H has the value (9) except that k is written for p_{3n-4}.

Now, since all the external forces acting on the system of $(n-1)$ particles act through the origin, therefore angular momentum round every line through the origin is conserved, and therefore the system has an invariable plane.

Take the plane of xy to be this invariable plane.

We have now two integrals of angular momentum, viz.

angular momentum about $Ox'=0$,

" " " $Oz'=-k\cos q_{3n-3}$

i. e.

$$p_{3n-3} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

$$-k\cos q_{3n-3} = \sum_{i=1}^{n-3} \begin{vmatrix} q_{3i-2}, & q_{3i-1} \\ p_{3i-2}, & p_{3i-1} \end{vmatrix} + \begin{vmatrix} q_{3n-3}, & q_{3n-1} \\ p_{3n-3}, & p_{3n-1} \end{vmatrix} + \begin{vmatrix} q_{3n-3}, & q_{3n-5} \\ p_{3n-3}, & p_{3n-5} \end{vmatrix}$$

$$\ldots\ldots\ldots (12).$$

Moreover, if H' denote the result of substituting these values of q_{3n-3} and p_{3n-3} for q_{3n-3} and p_{3n-3} in H, and if s be any one of the variables, then

$$\frac{\partial H'}{\partial s} = \frac{\partial H}{\partial s} + \frac{\partial H}{\partial q_{3n-3}} \cdot \frac{\partial q_{3n-3}}{\partial s}.$$

And

$$\frac{\partial H}{\partial q_{3n-3}} = \frac{dp_{3n-3}}{dt} = 0,$$

therefore

$$\frac{\partial H'}{\partial s} = \frac{\partial H}{\partial s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

The problem is now reduced to

$$\frac{dq_r}{dt} = \frac{\partial H'}{\partial p_r}, \quad \frac{dp_r}{dt} = -\frac{\partial H'}{\partial q_r} \quad (r = 1, 2, \ldots, 3n-5) \ldots (14),$$

where

$$H' = \sum_{i=1}^{3n-9} \frac{p_r^2}{2m_r} + \frac{1}{2m_{3n-6}}(p^2_{3n-8} + p^2_{3n-7} + \gamma^2) + \frac{1}{2m_{3n-3}}(p^2_{3n-6} + p^2_{3n-5} + \gamma'^2)$$

$$+ \frac{1}{2m_{3n}}\{(\sum_{i=1}^{n-3} p_{3i-2} + p_{3n-8} + p_{3n-6})^2 + (\sum_{i=1}^{n-3} p_{3i-1} + p_{3n-7} + p_{3n-5})^2$$

$$+ (\sum_{i=1}^{n-3} p_{3i} + \gamma + \gamma')^2\}$$

$$- \sum_{i,j=1}^{n-3} m_{3i} m_{3j} \{(q_{3i-2} - q_{3j-2})^2 + (q_{3i-1} - q_{3j-1})^2 + (q_{3i} - q_{3j})^2\}^{-\frac{1}{2}}$$

$$- m_{3n-6} \sum_{i=1}^{n-3} m_{3i} \{(q_{3i-2} - q_{3n-8})^2 + (q_{3i-1} - q_{3n-7}) + q^2_{3i}\}^{-\frac{1}{2}}$$

$$- m_{3n-3} \sum_{i=1}^{n-3} m_{3i} \{(q_{3i-2} - q_{3n-6})^2 + (q_{3i-1} - q_{3n-5})^2 + q^2_{3i}\}^{-\frac{1}{2}}$$

$$- m_{3n} \{\sum_{i=1}^{n-3} m_{3i} (q^2_{3i-2} + q^2_{3i-1} + q^2_{3i})^{-\frac{1}{2}} + m_{3n-3} (q^2_{3n-8} + q^2_{3n-7})^{-\frac{1}{2}}$$

$$+ m_{3n-3} (q^2_{3n-6} + q^2_{3n-5})^{-\frac{1}{2}}\}$$

$$- m_{3n-3} m_{3n-6} \{(q_{3n-6} - q_{3n-8})^2 + (q_{3n-2} - q_{3n-5})^2\}^{-\frac{1}{2}} \ldots\ldots\ldots\ldots (15),$$

where γ, γ' are given by a simplified form of equations (8), viz.

$$\gamma . q_{3n-7} + \gamma' . q_{3n-5} = -\sum_{i=1}^{n-2} \begin{vmatrix} q_{3i-1}, & q_{3i} \\ p_{3i-1}, & p_{3i} \end{vmatrix} \quad \ldots\ldots\ldots(16).$$

And $$\gamma . q_{3n-6} + \gamma' . q_{3n-4} = k \operatorname{cosec} q_{3n-3} + \sum_{i=1}^{n-3} \begin{vmatrix} q_{3i}, & q_{3i-2} \\ p_{3i}, & p_{3i-2} \end{vmatrix}$$

$$+ \cot q_{3n-3} \{-k \cos q_{3n-3}\},$$

where

$$-k \cos q_{3n-3} = \sum_{i=1}^{n-3} \begin{vmatrix} q_{3i-2}, & q_{3i-1} \\ p_{3i-2}, & p_{3i-1} \end{vmatrix} + \begin{vmatrix} q_{3n-6}, & q_{3n-7} \\ p_{3n-6}, & p_{3n-7} \end{vmatrix} + \begin{vmatrix} q_{3n-6}, & q_{3n-5} \\ p_{3n-6}, & p_{3n-5} \end{vmatrix},$$

i.e. $$\gamma . q_{3n-6} + \gamma' . q_{3n-4} = \sum_{i=1}^{n-3} \begin{vmatrix} q_{3i}, & q_{3i-2} \\ p_{3i}, & p_{3i-2} \end{vmatrix}$$

$$+ \left[k^2 - \left\{ \sum_{i=1}^{n-3} \begin{vmatrix} q_{3i-2}, & q_{3i-1} \\ p_{3i-2}, & p_{3i-1} \end{vmatrix} + \begin{vmatrix} q_{3n-6}, & q_{3n-7} \\ p_{3n-6}, & p_{3n-7} \end{vmatrix} + \begin{vmatrix} q_{3n-6}, & q_{3n-5} \\ p_{3n-6}, & p_{3n-5} \end{vmatrix} \right\}^2 \right]^{\frac{1}{2}}$$

$$\ldots\ldots\ldots(17).$$

§ 3. The system (14) of the $(6n-10)$th degree may (if desired) be further reduced to the $(6n-12)$th degree by the use of the integral of energy and the elimination of the time. For since γ, γ' involve p_{3n-9} only linearly, therefore H' is a quadratic function of p_{3n-9}, and therefore the equation

$$H' = h$$

can be solved in the form

$$p_{3n-9} + H_1 = 0.$$

According to the general theory, the equations (14) become

$$\frac{dq_r}{dq_{3n-9}} = \frac{\partial H_1}{\partial p_r}, \quad \frac{dp_r}{dq_{3n-9}} = -\frac{\partial H_1}{\partial p_r} \quad \left(\begin{matrix} r = 1, 2, \ldots, 3n-10, 3n-8, \\ 3n-7, 3n-6, 3n-5 \end{matrix} \right),$$

and the time is obtained by a quadrature from

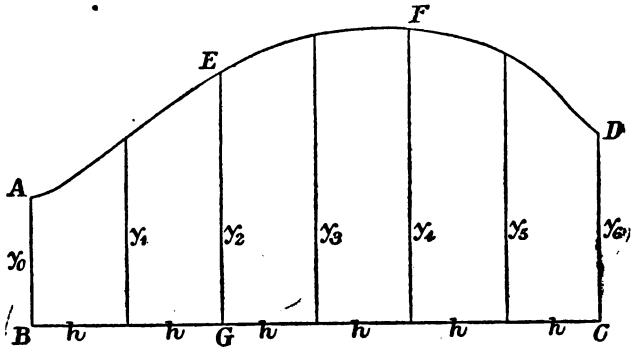
$$\frac{dt}{dq_{3n-9}} = \frac{\partial H_1}{\partial h}.$$

Trinity College,
Cambridge.

ON CERTAIN MODIFICATIONS IN THE COEFFICIENTS 1, 2, 4, 2, 4, ..., 2, 4, 1 USED IN SIMPSON'S RULE, WHEN EITHER OF THE OUTSIDE ORDINATES IS A TANGENT TO THE CURVED BOUNDARY OF THE FIGURE.

By *R. W. K. Edwards, M.A.*

"Simpson's Rule," it will be remembered, finds an approximation to the area of such a figure as *ABCD* in



terms of any odd number of equidistant ordinates and their common distance apart; the area being given in the case of seven ordinates by

$$S = \tfrac{1}{3}h\{y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6\}.$$

It seems to be often supposed* that the hypothesis on which Simpson's Rule should be based is that the successive portions of the irregular curved boundary, *AE*, *EF*, *FD*, approximate severally to being ordinary parabolic arcs. It can easily be seen, however, that if the arc *AE*, for instance, be supposed to approximate to any member of the family of curves

$$y = a + bx + cx^2 + dx^3,$$

the area of *ABGE* may be taken to be

$$\tfrac{1}{3}h\{y_0 + 4y_1 + y_2\},$$

and so for the other two portions. This gives a wider scope for the useful application of the rule than is often allowed.*

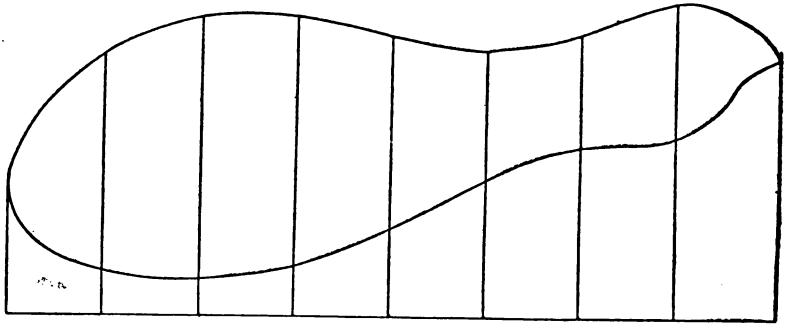
* See, for instance, the *Encyclopedia Britannica*, article *Mensuration*.

There may be, for instance, one point of contrary flexure in each of the arcs AE, EF, &c.

The rule cannot, however, be applied, with any expectation of close accuracy, to a curve in which there is a tangent perpendicular to the base BC of the figure to be quadrated. For at no point on the curve

$$y = a + bx + cx^2 + dx^3$$

can the gradient $\frac{dy}{dx}$ be infinite. This fact considerably vitiates the results obtained by applying either the common trapezoidal rule, or Simpson's or Weddle's rules, or any of the numerous formulæ which might be deduced by using the coefficients calculated by Cotes and given in Bertrand's *Calcul Intégral*, Section 363, in cases in which the outside ordinates (or one of them) touch the curved boundary, as in the figure.



There are many ways of attempting to reduce the inaccuracy so involved. A fresh set of coefficients may be calculated, on the same principle as in Simpson's Rule, but adjusting the ordinates so that those nearer the tangent ordinates are closer together than the others. The ratio of that part of the figure in which this kind of inaccuracy occurs to the rest is thus lessened. On the whole it seems best, however, to seek some new class of curves to which the boundary may be supposed to approximate in the neighbourhood of the point of tangency.

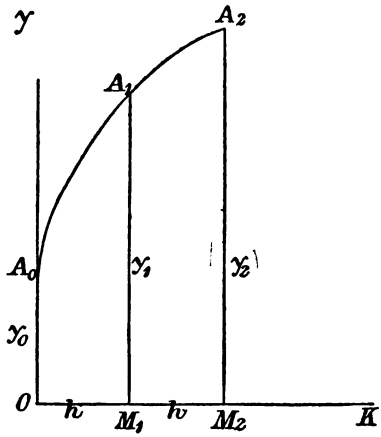
For the curves

$$y = a + bx^m + cx^n + \ldots$$

the gradient can nowhere be infinite at a point having finite coordinates unless at least one of the indices is fractional.

Also if the area is to be calculated in terms of three ordinates (the two outside and the ordinate half-way between them, say) there must not be in general more than three of the quantities a, b, c, ... to calculate. We may assign any values we please to m and n. It seems natural to give them the values 1 and $\frac{1}{2}$, and thus obtain the equation of an ordinary conic parabola.

Let y_0, y_1, y_2 be the first three of the equidistant ordinates, and let the curved boundary touch the ordinate OA_0 at A_0.



Suppose that the arc approximates to being a portion of the parabola

$$y = a + bx + c\sqrt{x},$$

which touches Oy at A_0, since $\frac{dy}{dx}$ will be infinite when $x = 0$.

Let the distance between successive ordinates be h; then the area $OA_0A_1A_2M_2$ will be approximately

$$\int_0^{2h} (a + bx + c\sqrt{x})\, dx,$$

i.e. $$2ah + 2bh^2 + \tfrac{2}{3}c\,(2h)^{\frac{3}{2}},$$

i.e. $$2ah + 2bh^2 + \frac{4\sqrt{2}}{3}\,ch^{\frac{3}{2}},$$

i.e. S, say.

Also $y_0 = a,$

$$y_1 = a + bh + ch^{\frac{1}{2}},$$

$$y_2 = a + 2bh + \sqrt{(2)}\, ch^{\frac{1}{2}}.$$

Now suppose that

$$\tfrac{1}{3}h\ (Ay_0 + By_1 + Cy_2) = S;$$

Then

$$\frac{A+B+C}{3}\, ah + \frac{B+2C}{3}\, bh^2 + \frac{B+C\sqrt{2}}{3}\, ch^{\frac{3}{2}}$$

$$= 2ah + 2bh^2 + \frac{4\sqrt{2}}{3}\, ch^{\frac{3}{2}}.$$

Equating the coefficients of a, b, c, we have

$$\frac{A+B+C}{3} = 2 \quad \text{.....(i)},$$

$$\frac{B+2C}{3} = 2 \quad \text{.....(ii)},$$

$$\frac{B+C\sqrt{2}}{3} = \frac{4\sqrt{2}}{3} \quad \text{.....(iii)}.$$

From (ii) and (iii)

$$\frac{(2-\sqrt{2})\ C}{3} = \frac{6-4\sqrt{2}}{3};$$

Hence

$$C = \frac{2\,(3-\sqrt{2})}{2-\sqrt{2}} = \frac{2\,(3-2\sqrt{2})\,(2+\sqrt{2})}{2} = 2-\sqrt{2}.$$

Also $B = 6 - 2C = 2\,(1+\sqrt{2}),$

and $A = 6 - (B+C) = 2-\sqrt{2}.$

Hence

$$S = \tfrac{1}{3}h\,\{(2-\sqrt{2})\,y_0 + 2\,(1+\sqrt{2})\,y_1 + (2-\sqrt{2})\,y_2\}.$$

The area of the next portion of the figure, between the third and fifth ordinates, will be, by Simpson's Rule,

$$\tfrac{1}{3}h\,\{y_2 + 4y_3 + y_4\};$$

and so on. Adding will give the approximation to the whole area as

$$\tfrac{1}{3}h\{(2-\sqrt{2})\,y_0 + 2\,(1+\sqrt{2})\,y_1 + (3-\sqrt{2})\,y_2 + 4y_3 + 2y_4 + 4y_5 + 2y_6 + \cdots\},$$

the last three coefficients being 2, 4, and 1, or $3-\sqrt{2}$, $2\,(1+\sqrt{2})$, and $2-\sqrt{2}$, according as the curve does not, or does, touch the last ordinate.

The values of the first three coefficients should, then, be modified to

·5858..., 4·284..., and 1·5858...

respectively.

Or, since $2-\sqrt{2} = 1-(\sqrt{2}-1)$,

$2\,(1+\sqrt{2}) = 4+2\,(\sqrt{2}-1)$,

and $3-\sqrt{2} = 2-(\sqrt{2}-1)$,

we may formulate a rule for finding the area when the first ordinate touches the curve thus:

Add together the first ordinate, four times the second, twice the third, ..., and so on, as in Simpson's Rule; then multiply the excess of twice the second ordinate over the sum of the first and third by $\sqrt{2}-1$, *i.e.* by ·4142; add the result to the first sum and multiply by one-third of the common distance between the ordinates.

For a complete oval this may be expressed: draw a pair of parallel tangents and an odd number of chords parallel to them, all the parallels thus drawn being equi-distant. Add together the lengths of the 2nd, 4th, ... chords and twice the lengths of the 1st, 3rd, ... chords; multiply the excess of the sum of the two outside chords over half the sum of the two chords next to them by ·4142; add the results and multiply by two-thirds of the common distance between the parallels.

As an instance of the practical effect of this modification of Simpson's Rule, it may be mentioned that the error in the area of a circle, by drawing a series of five parallel chords and employing Simpson's Rule, is 3 per cent.; while with the modified coefficients the error is reduced to ·4 per cent.

The choice of values for the indices m and n is entirely arbitrary. It would be unscientific to advance claims for accuracy for any special values, without some reference to the manner in which the alleged irregular curve to be quadrated is formed. Generations of users of Simpson's Rule have,

however, had reason to be content with the supposition that each item of arc approximated to an ordinary conic parabola; which seems some justification of the choice made above. Other choices can be made, and their comparative accuracy in the case of regular curves tested; but this tells us nothing as to the general accuracy for irregular curves.

Professor Alfred Lodge suggests the values $\frac{3}{2}$ and $\frac{1}{2}$ for m and n. These give $A=\frac{13-8\sqrt{2}}{15}$, $B=\frac{8\sqrt{2}}{15}$, $C=\frac{2}{15}$; and the accuracy of the results when applied to circles is remarkable. The values also allow of a point of contrary flexure in the arc; as is also the case if we suppose $m=\frac{2}{3}$, $n=\frac{1}{2}$, and get $A=\cdot160\ldots$, $B=1\cdot278\ldots$, $C=\cdot111\ldots$. On the other hand, the assumption of the ordinary conic parabolic arc makes A and C equal; and, moreover, is accurate in the case of a straight line (c being then equal to zero), which is likely to be useful when the figure to be quadrated is a segment of an oval curve.

A SPECIAL CONTINUANT EVALUATED BY CAYLEY.

By *Thomas Muir, LL.D.*

§ 1. In the *Quart. Journ. of Math.*, Vol. II., pp. 163–166, Cayley enunciated a theorem regarding the value of the continuant

$$\begin{vmatrix} \theta & 1 & . & . & \ldots & . & . \\ x & \theta & 2 & . & \ldots & . & . \\ . & x-1 & \theta & 3 & \ldots & . & . \\ . & . & x-2 & \theta & \ldots & . & . \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ . & . & . & . & \ldots & \theta & n-1 \\ . & . & . & . & \ldots & x-n+2 & \theta \end{vmatrix},$$

and gave a verification of the result. The paper is interesting for two reasons; first, because the process used by way of proof is neither simple nor elegant, as Cayley himself felt; second, because of the peculiar form of expression obtained,

viz. in terms of Sylvester's special continuants of the same type

$$\theta, \quad \begin{vmatrix} \theta & 1 \\ 1 & \theta \end{vmatrix}, \quad \begin{vmatrix} \theta & 1 & . \\ 2 & \theta & 2 \\ . & 1 & \theta \end{vmatrix}, \ \ldots$$

arranged in descending order. If we denote the original continuant by $U_n(x, \theta)$, these simpler continuants are

$$U_1(0, \theta), \quad U_2(1, \theta), \quad U_3(2, \theta), \quad \ldots,$$

and Cayley's identity is

$$U_n(x, \theta) = U_n(n-1, \theta) - \frac{n(n-1)}{2}(x-n+1)\, U_{n-2}(n-3, \theta)$$

$$+ \frac{n(n-1)(n-2)(n-3)}{2.4}(x-n+1)(x-n+3)\, U_{n-4}(n-5, \theta)$$

$$- \frac{n(n-1)\ldots(n-5)}{2.4.6}(x-n+1)(x-n+3)(x-n+5)\, U_{n-6}(n-7, \theta)$$

$$+ \ldots\ldots\ldots\

be desired, greater simplicity will be attained by further generalization. Writing Cayley's continuant, as continuants usually are written, with a minor diagonal of units, we have it in the form

$$\begin{vmatrix} \theta & 1.x & . & . & \dots \\ 1 & \theta & 2(x-1) & . & \dots \\ . & 1 & \theta & 3(x-2) & \dots \\ . & . & 1 & \theta & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}_n,$$

and see that it may be generalized by putting $x_1, x_2, x_3, \dots$ in place of the elements $1.x$, $2(x-1)$, $3(x-2)$, ... A little investigation then enables us to reach the following theorem: *If A_1 be written for the sum of the x's, A_2 for the sum of the products of two non-consecutive x's, A_3 for the sum of the products of three x's no two of which are consecutive, and so on, then*

$$\begin{vmatrix} \theta & x_1 & . & \dots \\ 1 & \theta & x_2 & \dots \\ . & 1 & \theta & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_n = \theta^n - A_1\theta^{n-2} + A_2\theta^{n-4} - A_3\theta^{n-6} + \dots \text{(I.)}.$$

Thus, when $n = 6$, the expansion is

$$\theta^6 - (x_1 + x_2 + x_3 + x_4 + x_5)\,\theta^4 + (x_1x_3 + x_1x_4 + x_1x_5 + x_2x_4 + x_2x_5 + x_3x_5)\,\theta^2 - x_1x_3x_5.$$

In Cayley's case of this the sum of the x's

$$= 1.x + 2(x-1) + 3(x-2) + \dots + (n-1)(x-n+2),$$
$$= \tfrac{1}{2}n(n-1)x - \tfrac{1}{3}n(n-1)(n-2),$$
$$= x.C_{n,2} - 2C_{n,3};$$

and the sum of non-consecutive pairs is found to be

$$x^2.3C_{n,4} - x.2(2n-5)C_{n,4} + \tfrac{4}{3}(5n-7)C_{n,5};$$

so that without going further his avoidance of this form of expansion is fully justified.

§ 3. The seeking for an expansion of the particular kind which he found is also readily justifiable, even without a knowledge of the result of the search. For, since Sylvester's

CONTENTS.

PAGE

On the reduction of the problem of n bodies. By T. L. BENNETT - - 113

On certain modifications in the coefficients 1, 2, 4, 2, 4, ..., 2, 4, 1 used in Simpson's Rule, when either of the outside ordinates is a tangent to the curved boundary of the figure. By R. W. K. EDWARDS - - - 121

A special continuant evaluated by Cayley. By THOMAS MUIR - - - 126

Papers for insertion should be sent to Dr. Glaisher, Trinity College, Cambridge, or to Messrs. Metcalfe and Co. Limited, Printing Office, Trinity Street, Cambridge.

An index to vols. i—xxv has been published. It is issued gratis to subscribers, but may be obtained separately, price Two Shillings.

No. CCCCV.] NEW SERIES. [January, 1905.

Stanford MAR 1905

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XXXIV.—NO. 9.

MACMILLAN AND CO.
London and Cambridge.
1905.

Price—One Shilling.

METCALFE AND CO. LIMITED, CAMBRIDGE.

continuant was known to be expressible in terms of $\theta^0, \theta^2, \theta^4, \ldots$ when of even order, and in terms of $\theta, \theta^3, \theta^5, \ldots$ when of odd order, it followed that θ^n must, for all positive integral values of n, be expressible in terms of $S_n, S_{n-2}, S_{n-4}, \ldots$, and that therefore any function, which like Cayley's continuant involved only the powers $\theta^n, \theta^{n-2}, \theta^{n-4}, \ldots$, must be expressible in terms of $S_n, S_{n-2}, S_{n-4}, \ldots$.

§ 4. When n is even the values of S_n may be written

$$S_2 = \theta^2 - 1,$$
$$S_4 = \theta^4 - (1^2 + 3^2)\,\theta^2 + 1^2.3^2,$$
$$S_6 = \theta^6 - (1^2 + 3^2 + 5^2)\,\theta^4 + (1^2.3^2 + 1^2.5^2 + 3^2.5^2)\,\theta^2 - 1^2.3^2.5^2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

and viewing this as a set of equations in $\theta^2, \theta^4, \theta^6, \ldots$ we obtain

$$\theta^2 = S_2 + 1,$$
$$\theta^4 = S_4 + (1^2 + 3^2)\,S_2 + 1,$$
$$\theta^6 = S_6 + (1^2 + 3^2 + 5^2)\,S_4 + (1^2.3^2 + 1^2.3^2 + 3^2.5^2)\,S_2 + 1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

and generally, if we put

$$(a, b, c, d, \ldots)^r$$

for the sum of the r-ary products of $a, b, c, \ldots$, *i.e.* for

$$\Sigma a^r + \Sigma a^{r-1}b + \Sigma a^{r-2}b^2 + \Sigma a^{r-2}bc + \ldots,$$

we have

$$\theta^{2m} = S_{2m} + \{1^2, 3^2, 5^2, \ldots, (2m-1)^2\}^1\, S_{2m-2} + \{1^2, 3^2, 5^2, \ldots, (2m-3)^2\}^2\, S_{2m-4} + \{1^2, 3^2, 5^2, \ldots, (2m-5)^2\}^3\, S_{2m-6} + \ldots + 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\

§ 5. This arbitrary separation of Sylvester's continuants into those expressible in terms of even powers of θ, and those expressible in terms of odd powers, is the main cause of the complexity which appears in Cayley's paper. To do away with the possibility of such separation let us substitute $\theta + n$ for θ in S_n, calling the result S_n'. We then have

$$S_1' = \theta + 1, \quad S_2' \equiv \begin{vmatrix} \theta+2 & 1 \\ 1 & \theta+2 \end{vmatrix} = (\theta+1)(\theta+3),$$

$$S_3' \equiv \begin{vmatrix} \theta+3 & 1 & . \\ 2 & \theta+3 & 2 \\ . & 1 & \theta+3 \end{vmatrix} = (\theta+1)(\theta+3)(\theta+5), \ \ldots.$$

From this single set of equations it is easy, in the manner of the preceding paragraph, to express θ^n in terms of

$$S_n', \ S'_{n-1}, \ S'_{n-2}, \ \ldots,$$

the result in fact being

$$\theta^n = S_n' - (1, 3, 5, \ldots, 2n-1)^1 S'_{n-1} + (1, 3, 5, \ldots, 2n-3)^2 S'_{n-2} - \ldots + (-1)^n \ldots (\text{IV.}).$$

§ 6. Treating Cayley's continuant in like fashion, viz. putting $\theta + x + 1$ for θ and calling the result C_n', we obtain as a substitute for (α) of § 1

$$C_2' \equiv \begin{vmatrix} \theta+x+1 & 1 \\ x & \theta+x+1 \end{vmatrix}$$

$$= (\theta+1)(\theta+3) + 2(x-1)(\theta+1) + x(x-1),$$

$$C_3' \equiv \begin{vmatrix} \theta+x+1 & 1 & . \\ x & \theta+x+1 & 2 \\ . & x-1 & \theta+x+1 \end{vmatrix}$$

$$= (\theta+1)(\theta+3)(\theta+5) + 3(\theta+1)(\theta+3).(x-2) + 3(\theta+1).(x-1)(x-2) + x(x-1)(x-2);$$

$$C_4' \equiv (\theta+1)(\theta+3)(\theta+5)(\theta+7) + 4(\theta+1)(\theta+3)(\theta+5).(x-3) + 6(\theta+1)(\theta+3).(x-2)(x-3) + 4(\theta+1).(x-1)(x-2)(x-3) + x(x-1)(x-2)(x-3);$$

and generally

$$C_n' = S_n' + \binom{n}{1}(x-n+1)\,S'_{n-1}$$

$$+ \binom{n}{2}(x-n+2)(x-n+1)\,S'_{n-2} + \ldots \quad \ldots(\text{V.}).$$

§ 7. The fact that C_n degenerates into S_n, and C_n' into S_n', when $x=n-1$, explains in part the form of the development found in (α) and in (V). It also suggests consideration of another equally interesting continuant, viz.

$$\begin{vmatrix} \theta & x-n+2 & . & \ldots & . & . \\ x & \theta & x-n+3 & \ldots & . & . \\ . & x-1 & \theta & \ldots & . & . \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ . & . & . & \ldots & \theta & x \\ . & . & . & \ldots & x-n+2 & \theta \end{vmatrix} \equiv M_n, \text{ say,}$$

in which the reversing of the one minor diagonal gives the other.

For the same reason as in the case of C_n it must be expressible in terms of S_n, S_{n-2}, S_{n-4}, ... Further, since like C_n it changes into S_n for $x=n-1$, and has the additional peculiarity of doing the same for another value of x, viz. $x=-1$, it follows that the difference between it and S_n must be a multiple of $(x-n+1)(x+1)$. We are thus prepared for the following theorem analogous to Cayley's:

$$M_n = S_n - \binom{n-1}{1}.(x-n+1)(x+1).S_{n-2}$$

$$+ \binom{n-2}{2}.(x-n+1)(x+1).(x-n+2)x.S_{n-4}$$

$$- \binom{n-3}{3}.(x-n+1)(x+1).(x-n+2)x.(x-n+3)(x-1).S_{n-6}$$

$$+ \ldots \quad \ldots\ldots\ldots\

QUADRATIC PARTITION TABLES.—ERRATA.

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's Coll., London.

[The thanks of the Author are due to Mr. H. J. Woodall, A.R.C.Sc., for kindly verifying the Errata in Reuschle's Tables, and for reading the proof sheets.]

In the course of compilation of a large set of Tables of *Quadratic Partitions** (recently published), the author was led to collate the new Table with earlier similar Tables with the view of detecting errors and omissions. It seems worth while now to publish a list of those found in the earlier Tables.

The Tables in question give the *quadratic parts* (a, b), (c, d), &c., of prime numbers (p), as stated below, to argument p.

Quadratic parts,	(a, b);	(c, d);	(A, B);	(L, M);	(X, Y).
Partition,	$p = a^2 + b^2$;	$c^2 + 2d^2$;	A^2+3B^2;	$\frac{1}{4}(L^2+27M^2)$;	$\frac{1}{4}(X^2-5Y^2)$.

The authorship, date, and nature of the Tables examined are briefly indicated below :

1. *Jacobi's*, 1846; (a, b), (c, d), (A, B).
2. *Reuschle's*, 1856; (a, b), (c, d), (A, B), (L, M).
3. *Lloyd Tanner's*, 1873; (X, Y).
4. *Cunningham's*, 1904; All the above (and many others).

Here follows a brief description of these Tables, ending in each case with a List of the *Corrigenda* required in each. It has been thought sufficient to give in most cases only the *Corrigenda* without printing the actual *Errata* (except when the Argument—which is usually p—is itself the Erratum or omission).

APPENDIX.

(1). *C. G. J. Jacobi's* Tables, published in *Crelle's Journal*, t. xxx., 1846, pages 174—80; and again in Jacobi's *Opuscula Mathematica*, Bd. i., Berlin, 1846, pages 326—332. These give (a, b), (c, d), (A, B) as follows:

		Crelle, pp,	*Jacobi*, pp.
Tab. I.	gives (a, b) up to $p \not> 12000$;	174—176;	326—328.
Tab. II.	gives (A, B) up to $p \not> 12000$;	177—179;	329—331.
Tab. III.	gives (c, d) up to $p \not> 6000$;	180;	332.

Tab. I., II. were computed by *Herr Zornow*, Tab. III. by *Dr. Struve*. Here follows a List of the *Corrigenda* required.

	Table I of (a, b)	Tab. II of (A, B)
Read	p = 2357, 3253, 3469, 3509, 5693;	3631, 6427;
Instead of	2457, 2253, 3459, 2529, 5093;	2631, 6433;

* *Quadratic Partitions* (by the present author), published by Francis Hodgson, 89 Farringdon Street, London E.C.

Omissions, Tab. I.			*Omissions*, Tab. II.			*Corrigenda of* a, b			*Corrigenda of* A, B		
p	a	b	p	A	B	p	a	b	p	A	B
197	1	14	883	4	17	5261	19	70	6481	41	40
2713	3	52	6427	80	3	8609	47	80			
6997	39	74	11311	106	5						
11173	97	42									

(2). *C. G. Reuschle's Neue Zahlentheoretische Tabellen, &c.*, Stuttgart, 1856. These give (a, b), (c, d), (A, B), (L, M) as follows:—

Tab. IIIa, pp. 23—25, gives (A, B), (L, M) up to $p = 5743$.

Tab. IIIb, pp. 26—28, continues (A, B), up to $p \not> 13670$.

Tab. IIIc, pp. 29—32, continues (A, B), [when $(10/p)_3 = 1$] up to $p \not> 50{,}000$.

[In Tab. IIIa, b, the primes, which give $(10/p)_3 = 1$, are marked with an *].

Tab. IVa, pp. 32—38, gives (a, b) up to $p \not> 12400$ and (c, d), [when $p = 8\varpi + 1$], up to $p \not> 12400$.

Tab. IVb, pp. 38—41, continues (a, b), [when $(10/p)_2 = +1$], up to $p \not> 25000$.

Tab. IVc, p. 41, continues (c, d), [when $p = 8\varpi + 1$, and $(10/p)_4 = +1$], up to $p \not> 25000$.

[The primes which give $(10/p)_4 = +1$ are marked with an * in Tab. IVa, b those which give $(10/p)_8 = +1$ are marked with *two* ** in Tab. IVa, and with *one* * in Tab. IVc].

These Tables were apparently intended to serve chiefly for testing the 3ic., 4ic., and 8ic. residuacity of the base 10; the restrictions on the form of p in Tab. IIIc, IVa, b, c cause these to be very incomplete Tables of (A, B) beyond $p > 13670$, and of (a, b) beyond $p > 12400$ and of (c, d) throughout.

The Tables are unfortunately badly† printed, and contain many‡ Errata. The Errata are of *two* kinds, *viz.* (1) those affecting the quadratic partitions, (2) those relating to the residuacity of the base 10; these latter appear to be partly due to the fact that the Rule stated§ by Reuschle for discriminating between $(10/p)_4 = \pm 1$, is *not really accurate*; his application of the criterion for $(10/p)_8 = \pm 1$ seems also to be in some way‖ faulty.

† In some places the type is so indistinct as to be liable to be misread; it is quite possible therefore that some of the figures, here reported among the Errata, were really correct in the original.

‡ A good many of these Errata had been previously discovered, and communicated to the present writer, by the late Mr. Chas E. Bickmore

§ Reuschle's Rule (as stated on p. 6 of of his Memoir quoted) requires to be supplemented by the preliminary condition that $(10/p)_2 = +1$; see. C. E. Bickmore's Paper *On the numerical factors of* $(a^n - 1)$, *Second Notice*, Art. 3, in *Messenger of Mathematics*, Vol. 26, 1897.

‖ Thus Table IVc contains only 79 entries: among these there are 19 errata in the assignment of the asterisk (*) defining whether $(10/p)_8 = \pm 1$.

Here follows a List of the *Corrigenda* required.

Tab. IIIa, b, c of (A, B), (L, M), pages 23—32. Errata as below. [Arg. *p*.]

Page 32. Head-line of Column 1; ***Read*** *p*, A, B, ***instead of*** *p*, B, A.

Pages 26, 31. ***Omit*** (the non-primes) 6433, 41197 with their A, B.

Pages 26, 27. ***Insert*** an asterisk (*) after the primes 8167, 8317.

Pages 29—32. ***Omit*** the primes 16561, 18301, 18481, 23557, 35257, 45307, 45361 with their A, B, as $(10/p)_2 \neq 1$.

Corrigenda in p†		*Insert omissions*						*Corrigenda in* L, M.		
Read	Instead of	*p*	A	B	*p*	A	B	*p*	L	M
17137	17136	883	4	17	25453	95	74	139	23	1
25189	25183	11311*	106	5	25747	160	7	397	34	4
25579	5579	12553	101	28	27631	166	5	1123	35	11
26479	26459	12739*	8	65	32353	175	24	2377*	79	11
30703	30763‡	12967*	110	17	33037	65	98	2713	103	3
31051	51051	12973*	65	54	34519	38	105	4003	107	13
32353	32553	12979	76	49	35437	65	102	4339	128	6
40759	40659	13477	107	26	37699	68	105	5437	146	4
49279	49277	13537	113	16	39181	191	30	5503	148	2
		19891	104	55	43201	1	120			
		20443	100	59	44563	200	39	*Omission*		
		21499	68	75				883	47	7

Corrigenda in A, B.

p	A	B	*p*	A	B	*p*	A	B
313	11	8	18427	100	53	27691	104	75
5011*	56	25	18481	127	28	29059	128	65
5653	19	42	18553	35	76	29179	152	45
8293§	91	2	19423	130	29	30529	23	100
8707	92	9	19477	35	78	35257	53	104
9871§	38	53	20071	86	65	37363	20	111
10957	47	54	21391	146	5	37507	160	63
12211*	56	55	22651	76	75	38449	193	20
12823	106	23	23557	37	86	45307	212	11
16561	127	12	25147	140	43	45361	193	52
18301	7	78	26317	145	42			

Tab. IVa, b, c *of* (a, b), (c, d), *pages* 32—41. *Corrigenda.*

Primes misprinted.† Tab. IVa, page 34; Read 3469 for 3459.

Tab. IVb, page 40; Read 23893 for 29893.

Primes wrongly inserted. Tab. IVb, pages 39, 40; Omit 12697, 16981, 19381, 21101 with their a, b as $(10/p)_2 = -1$.

Tab. IVc, page 41; Omit 16649 with its c, d, as $(10/p)_4 = -1$.

† The Errata in *p* seem to be typographical errors; as the primes, when corrected, will be found in their right places.

‡ The 30763 in top-line of page 30 should be 30703; the 30763 in third line is correct.

§ The A, B of 8293 and 9871 are correct in some copies.

Asterisks omitted or superfluous.

Tab. IVa, b, pages 33—41. Insert *one* * after $p = 733, 2213, 2477, 2677, 2729, 3169, 3373, 6997, 11117, 14293, 14929, 17317, 20357, 21613, 21649, 22277, 23293, 24733.$

Tab. IVa, pages 34—38. Insert *two* ** after $p = 2161, 12289.$

Tab. IVa, b, pages 33—40. Omit the * after $p = 1213, 2437, 16649, 22093.$

Tab. IVa, pages 34—38. Omit *one* * after $p = 2129, 6761, 7561, 8521, 8689, 11801, 12329.$

Tab. IVc, page 41. Insert *one* * after $p = 14081, 15601, 15641, 15761, 17489, 17729, 19489, 24809, 24889.$

Tab. IVc, page 41. Omit the * after $p = 13729, 14321, 15361, 16249, 17209, 17449, 18329, 19289, 20681, 23561.$

Tab. IVa, b, pages 32—41						Tab. IVa, pp. 32-38			Tab. IVc, page 41		
Omissions			*Corrigenda in* a, b			*Corrigenda in* c, d			*Corrigenda in* c, d		
p	a	b	p	a	b	p	c	d	p	c	d
197	1	14	4421	65	14	17	3	2			
11173	97	42	14009	115	28	1777	25	24	14009	69	68
12269	13	110	15361	31	120	4177	55	24	14081	117	14
12301	99	50	16249	43	120	6553	55	42	14369	111	32
12373	103	42	17317	129	26	6653	—	—	14929	121	12
12973	83	78	18289	135	8	7481	57	46	17489	99	62
15493	97	78	19489*	105	92	8969	63	50	17729	111	52
16253	37	122	21613	147	2	11057	105	4	19001	123	44
17077	119	54	23197	101	114	11113	49	66	19489	133	30
17117*	91	94	23561	131	80	11329	31	72	23929	139	48
17929	125	48	24281	155	16	12049	41	72			
18517	119	66				12097	107	18	*Omission*		
21493*	87	118				12161	63	64	22129	77	90
22129*	15	148				12281	27	76			

(3). *H. W. Lloyd-Tanner's* Paper *On the Complex primes formed with the Fifth Roots of Unity* in *Proc. Lond. Math. Soc.*, Vol. xxiv., 1873, gives (pages 256—262) a Table of the *Least Solutions* of $p = \frac{1}{4}(X^2 - 5Y^2)$ up to $p \not> 10{,}000$.

Omission (page 258); $p(X, Y) = 3371, (127, \dot{2}\dot{3})$.

(4). *Lt.-Col. Allan Cunningham's Quadratic Partitions*, 1904. These give *all* the quadratic partitions of the earlier Tables (Nos. 1, 2, 3), mostly up to much higher limits, viz. :—

(a, b), (c, d), (A, B), (L, M) up to $p \not> 100{,}000$; (X, Y) up to $p \not> 10{,}000$; besides many other partitions. A few Errata,* discovered whilst the work was still in the press, are printed on the back of the Title-page. Also,

* The author would be glad to receive a Note of any more Errata discovered; these may be sent to him, addressed to the care of the publishers.

most of the Errata above detailed in the earlier Tables (excluding those in Heuschle's Tables relating to the residuacity of 10) are *mentioned* in an Appendix (pages 265, 266); the present List is, however, much more complete. The following misprints occur in that Appendix:

List of p omitted. Read $p = 27631$, not 27431.
List of (A, B) *incorrect.* Read $p = 5653$, not 5453.
List of (c, d) *incorrect.* Read $p = 11113$, not 11153.
Sixth line from foot. Read page 42, not page 43.

Here follow *Corrigenda* on the "Introduction," (pages i.—xxiii.)

page	*line*	*For*	*Read*	*page*	*line*	*For*	*Read*
i.	10 up	$v^3 + w^3$	$v^3 + 11w^3$;	xx.	16	2^{24}	5^{24}
xiv.	6 up	300024	30024 ;	xxi.	17	(A, B)	(A, B), (L, M)
xvi.	9	-9604^3	-5.9604^3;	xxi.	7 up	causes	cause

CURVES ON A CONICOID.

By *Harold Hilton.*

§ 1. LET Σ be any conicoid with two systems of generators Γ and Γ'; any n-ic (algebraic curve of degree n) on Σ meeting each generator of the system Γ in h points, and each generator of the system Γ' in k points $[n = h + k]$ will be called an $(h + k)$-ic. The generators of the systems Γ, Γ' through any point A on Σ will be denoted by α, α' respectively, and so on.

§ 2. Suppose we project figures on Σ on to any plane Π from a point A on Σ. Let α, α' meet Π in i, i'. Then all generators of the systems Γ, Γ' project into straight lines through i' and i respectively.

Let Φ be any conic on Σ, and ϕ its projection on Π. Then ϕ passes through the projections on Π of the points I, I' in which Φ cuts α and α'. Hence the projection of any conic* on Σ passes through i and i'.

Let O be the pole of the plane of Φ with respect to Σ, and o the projection of O on Π. Then since the planes AIO, $AI'O$ touch Σ at I and I', the lines oi, oi' touch ϕ at i and i'; *i.e.* o is the pole of ii' with respect to ϕ.

If any line through O meets Σ in P and Q, the projections p, q, of these points on Π are collinear with o and conjugate

* Unless the conic passes through A, when its projection is a straight line.

with respect to ϕ. For if the plane $AOPQ$ cuts Φ in C, D and ϕ in c, d, the pencil $A(PQ, CD)$ is harmonic (since OC, OD are the tangents at C and D to the conic $ACDPQ$), and therefore the range (pq, cd) is harmonic.

§ 3. Any $(h+k)$-ic Θ on Σ projects in general into a plane n-ic θ with an h-ple point at i and a k-ple point at i'. The intersections of Π with the tangent planes to Σ at the points in which the generators α and α' meet Θ are the tangents to θ at i and i'. If, however, A is an r-ple point on Θ, θ is an $(n-r)$-ic with an $(h-r)$-ple point at i and a $(k-r)$-ple point at i' meeting the line ii' in r other points.

The tangents at these r points are the intersections of Π with the planes osculating Θ at A. If Θ has a cusp at A, θ touches ii'. If α touches Θ, two of the tangents to θ at i coincide. If Θ touches α at A, θ is an $(n-1)$-ic touching ii' at i with an $(h-1)$-ple point at i and a $(k-1)$-ple point at i'. If Θ touches a generator of the system Γ' at its point of intersection with α, θ has an inflexional tangent at i. If two curves touch at a point on α their projections osculate at i.

§ 4. The following theorems illustrate the method; they may be proved by projecting from A (or otherwise).

A generator of the system Γ cuts a $(2+1)$-ic at A, B, and two generators of the same system touch the curve at L, K and cut β' at M, N. The conics ALN, AKM on Σ touch at A; β' meets the intersection of the plane ALK with the osculating plane at A; λ', κ', α', β' meet any generator of the system Γ' in a harmonic range; if conics on Σ touching the curve at A and B intersect on λ', they meet again on κ'.

If the four generators which touch a $(3+1)$-ic meet the curve again at A, B, C, D, the three conics on Σ touching the curve at B, C, D and passing through A are coaxial.

If any conic through two fixed points A, B on a $(3+1)$-ic meets the curve again at R and S, ρ' and σ' cut any fixed generator of the system Γ in a pair of an involution range.

If α touches a $(2+2)$-ic with a cusp B, and the osculating planes at P and Q pass through A, the intersection of the bitangent plane through A and the plane APQ meets β'.

If the osculating planes of a $(3+2)$-ic at P and Q pass through a node A, and the plane APQ meets the curve again at R, the osculating plane at R passes through A.

If a $(4+2)$-ic has a node A and cusps H, K, and if α touches the curve at P, π' meets the intersection of planes through A, touching the curve at H and K.

§ 5. Let the $(h+k)$-ic Θ with δ nodes and κ cusps be projected into the plane π-ic as in § 3. Then the number of tangent planes to Θ through two fixed points = the number of tangents to θ from any given point

$$=(h+k)(h+k-1)-h(h-1)-k(k-1)-2\delta-3\kappa=2hk-2\delta-3k.$$

The maximum number of double points of Θ is

$$\tfrac{1}{2}[(h+k-1)(h+k-2)-h(h-1)-k(k-1)]=(h-1)(k-1).$$

Using the notation of § 2, let p be any point on θ and q a point on op such that p and q are conjugate with respect to ϕ. Let Λ be the locus of the pole of ii' with respect to the conic $piqi'$ touching θ at p (ϕ is fixed, but p varies), *i.e.* the envelope of the third diagonal (other than ii' and pq) of the quadrangle $piqi'$. Then to each of the n points in which any given straight line through o meets θ corresponds a tangent to Λ from the pole of this line with respect to ϕ. To each tangent from o to θ corresponds an intersection of Λ with ii'. To each cusp of θ corresponds an inflexion of Λ. Hence Λ has the same Plücker's numbers as the polar reciprocal of θ. Now to each node or cusp of Λ corresponds a conic bitangent to or osculating θ and passing through points collinear with o and conjugate with respect to ϕ; therefore to each node or cusp of Λ corresponds a plane through O bitangent to or osculating Θ.

Hence, if an $(h+k)$-ic on a conicoid has δ nodes and κ cusps, and is such that m tangent planes pass through any two points, and such that τ bitangent planes and ι osculating planes pass through a given point (which may, or may not, lie on the conicoid)

$$m=2hk-2\delta-3\kappa,$$

$$h+k=m(m-1)-2\tau-3\iota,$$

$$\iota=3(2hk-h-k)-6\delta-8\kappa,$$

$$\kappa=3m(m-2)-6\tau-8\iota,$$

$$\text{deficiency}=(h-1)(k-1)-(\delta+\kappa)=\tfrac{1}{2}(m-1)(m-2)-(\tau+\iota)\not<0.$$

Such a curve is determined when $(hk+h+k-\delta-2\kappa)$ points are given. The number of conics of four-point contact is $(3\iota-3m+n)$.

§ 6. If $h=k=\frac{1}{2}n$, Θ is the complete intersection of Σ with an algebraic surface of degree $\frac{1}{2}n$.* If Σ has imaginary generators, all real algebraic curves on Σ must be of this type.

* See *Mathematical Gazette*, July 1904, p. 60.

When this surface is a cone the properties of Θ are most conveniently investigated by projecting points on Σ from the vertex V on to any plane. All conics* on Σ project into conics having double contact with a fixed conic j (the projection of the intersection of Σ with the polar plane of V), and all generators of Σ project into tangents to j. For illustrations, see *Proc. Lond. Math. Soc.*, 2, ii., pp. 155–159.

NOTES ON THE GEOMETRY OF CUBIC SURFACES.

The following notes were suggested by the article *Geometry* in the *Encyclopædia Britannica.*

By *W. H. Blythe, M.A.*, Jesus College.

The intersection of three corresponding planes of three projective pencils lies on a cubic surface.—[Reye].

It is interesting to test this statement by analysis. If we take two points S and S', and join them to every point of a plane P, not containing S or S', we get two pencils of lines or rays.

These pencils are said to be *perspective.* Corresponding rays are those joining the same point of P to S and S'. Corresponding planes are those joining the same straight lines in P to S and S'.

If one of these pencils be altered in position, the rays of the pencil still keeping the same position with respect to one another, the pencils are said to be *projective.*

Now take Cartesian coordinates, let $Lx + My + Nz = 0$ be the equation to any plane through the origin S. Transfer the pencil of rays to S' turn them about through any angle, the equation to the plane will now be $LX + MY + NZ = 0$, where X, Y, Z are functions of x, y, z of the first degree.

Again transfer the pencil to S'' a third fixed point, the equation to the plane now becomes $LX' + MY' + NZ' = 0$, where X', Y', Z' are also functions of x, y, z of the first degree. Eliminate L, M, N, we have

$$x(YZ' - ZY') + y(ZX' - XZ') + z(XY' - YX') = 0,$$

which is the equation to a cubic surface through S, S', and S''.

Corresponding rays from two pencils do not always meet, but when they do the point lies upon the cubic surface independently of the third pencil, for the equations to corresponding

* Unless their planes pass through V, when they project into straight lines.

rays would be $lx = my = nz$ and $lX = mY = nZ$, in which case $x : y : z :: X : Y : Z$. This point lies on a twisted cubic joining S and S'.

The generation of a twisted cubic by two projective pencils is explained geometrically in the *Encyclopædia Britannica* under the heading Geometry, see the 9th edition, p. 389, paragraph 101 to 104. It is there shown also that two pencils are projective when four elements, four rays, or four planes correspond, paragraph 92.

Following the same line of argument we can show that any three points on the surface may be used as the centres of projective pencils to generate the surface, provided they are not in the same straight line.

Generally, also a straight line meets the surface in one or three points, or more exactly in three points, two of which may be imaginary. [*Geometry of Position*, Theodor Reye, translated by Thos. Holgate, p. 141.]

That a straight line does not generally meet the surface in more than three points may be inferred thus:

Suppose S and S' are two points on the surface, then SS' does not again meet the twisted cubic through S, S', but it meets the surface again where the corresponding plane from S'' meets it, for the straight line SS' is the intersection of two corresponding planes from S and S'. A straight line lies entirely in the surface when the three corresponding planes from S, S', S'' meet along a straight line instead of intersecting at a point.

We must not have four non-intersecting straight lines on the surface that also lie on a conicoid, for in this case it may be shown that every straight line on the conicoid has four points common with the cubic surface, and lies entirely upon it. A cubic surface may be determined by three arbitrary points S, S', S'' upon the surface, and four straight lines upon the surface, for by joining S, S', S'' to the four lines we fix four elements upon each pencil.

We can now show that if we exclude the case of ruled surfaces, and also those in which the surface degenerates into a conicoid and plane, that there are twenty-seven straight lines on the surface.

First we have six non-intersecting straight lines, each the common chords of two twisted cubics, for the three pencils taken two and two.

Next any five of these meet one straight line, for if not by taking the six straight lines four at a time, and taking two straight lines to meet each set of four, we obtain six straight

lines on the surface that meet three non-intersecting straight lines, and therefore lie on a conicoid, but this, as we have stated, is impossible.

Since each set of five meets a straight line, this straight line lies on the surface, and we get six more non-intersecting straight lines on the surface, forming a double six with the first set.

Every plane containing two intersecting straight lines of the double six must meet the surface again in another straight line, determined by joining the points at which the plane meets the remaining lines of the double six, for it is evident that these straight lines have four points common with the surface. These straight lines altogether make up twenty-seven.

It can easily be shown that there cannot be more than twenty-seven, for if so we must arrive at four straight lines in one plane section, which is impossible.

To prove by Pascal's Theorem that the straight lines meeting three non-intersecting straight lines generate a conicoid, i.e. a surface, every plane section of which is a conic.

[*Construction.* Let $ABCDEF$ be a plane hexagon, so that the opposite sides AF, CD; FE, BC; ED, AB meet respectively in L, M, N.]

Let P_1, P_2, P_3; p_1, p_2, p_3 be two sets of non-intersecting straight lines in space, each line of one set meeting all of the other set, let them meet any plane in the points A, C, E, B, D, F. The plane is not to contain one of the given straight lines.

Denote the intersections P_1p_2, P_2p_3, P_3p_1 by X, Y, Z. Now X, Y, L lie in one straight line, for they will all be found in both planes P_1p_3, and P_2p_3, and lie on their intersection. Similarly, Y, Z, M lie in the planes P_2p_3 and P_3p_1, and X, Z, N in P_3p_3 and P_1p_1.

Therefore L, M, N are the points in which the sides of the triangle XYZ meet the plane, and are therefore in one straight line, so that A, B, C, D, E, F lie on a conic.

Since five points determine the conic we may consider F and therefore p_3 as variable, and any straight line meeting P_1, P_2, P_3 traces out the conic $ABCDE$.

Cor. Only two straight lines can be drawn to meet four non-intersecting straight lines P_1, P_2, P_3, P_4.

Construct the two conicoids through P_1, P_2, P_3, and P_2, P_3, P_4. These meet any plane in two conics, two points of intersection of which are known, the other two determine two common generators.

ON A PAPER-FOLDING PUZZLE.

By *H. M. Taylor.*

SOME few months ago a friend of mine attracted my attention to the puzzle "How to fold a sheet of paper in which a circular hole, exactly the size of a shilling, has been cut out, so that a half-crown would go through the hole." My friend said he had done the puzzle, but he wanted an explanation of the reason why it was possible to do a thing which at first sight seemed to be impossible.

The apparent impossibility, to my mind, rests on the fact that paper is to all intents and purposes almost unstretchable.

It is true that no perfectly inextensible surface, no matter how flexible it may be, can, without destroying the continuity of the surface, be deformed in such a way as to increase the length of any continuous line drawn on the surface. In so far as paper is inextensible, it is impossible to deform it in such a manner as to increase the length of the straight line joining two points on the surface, provided that there is no discontinuity between the points.

It is different, however, if there be a hole or a slit in the paper between the points.

Let us take a triangular piece of paper ABC, out of which a triangular piece AOB is cut away. Then let us draw on the surface of the paper two straight lines OX, OY containing an angle XOY equal to half the supplement of the angle AOB. Then, if we crease the paper in opposite directions along the lines OX, OY, and fold the paper flat down on itself, the two lines AO, OB will form parts of the same straight line, the distance between the two points A, B being by this process increased.

If the process which has just been described be called "putting in a pleat," we may say the two sides of an open triangular hole in a piece of paper may be brought into line by putting in one pleat, the three sides of an open quadrilateral hole by putting in two pleats, and so on. An infinite number of infinitesimal pleats would be required to straighten the boundary of an open curvilinear hole.

Next let us consider the case of a closed hole. First, for simplicity, we take the case of a piece of paper out of which a *closed* square hole $ABCD$ has been cut. First crease the paper along the line AC produced both ways beyond A, C; then fold the paper flat down on itself, bringing the points B, D into conjunction; next put into the doubled paper at the two points B, D a pleat having an angle equal to half a right angle.

This operation will bring the four sides of the square into a doubled straight line. It will, however, be found that the two halves of the paper into which it is divided by the straight line *A*, *C* are engaged at the points *B*, *D*.

If the paper be now opened out and then folded down flat on itself again, taking care to keep the two halves disengaged, it will be found that there is a continuous opening between the two points *A*, *C* equal to twice the side of the square.

Figure 1 gives a somewhat neater way of folding a paper so as to bring the sides of a square into line.

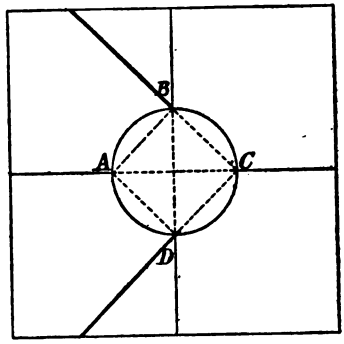
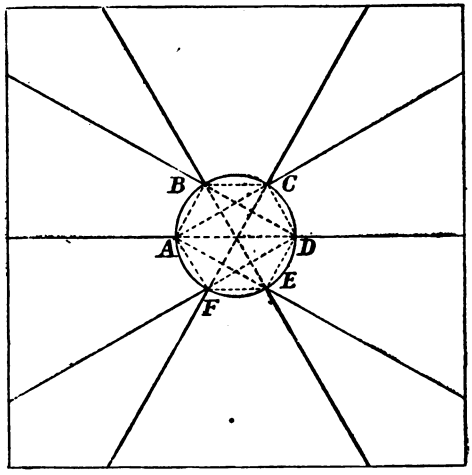


Figure 2 gives a symmetrical method of folding paper so as to bring the six sides of a regular hexagon into line. In



both figures the bending is to be made in opposite directions along the lines marked with heavy and light lines.

NOTE ON A PROPOSITION STATED BY SCHLESINGER.

By *E. Cunningham.*

In Schlesinger's *Theorie der Linearen Differentialgleichungen*, Bd. I. § 95, the following proposition is given, and, so far as I can ascertain, it has not been challenged.

If $z^{(n)} + \phi_p(x)\, z^{(n-1)} + \phi_{2p}(x)\, z^{(n-2)} + \ldots + \phi_{np}(x)\, z = 0$

be a linear differential equation in which $\phi_r(x)$ denotes a polynomial of degree r, a fundamental set of solutions of the equation is given by

$$z_\lambda = e^{\{c_{0\lambda} x^{p+1} + c_{1\lambda} x^p + \ldots + c_{p\lambda} x\}}, \quad \{\lambda = 1 \ldots n\},$$

the quantities 'c' being suitably determined constants.

That this proposition is not generally true may be easily seen by taking an equation of the second order with $p = 1$,

$$z'' + (ax + b)\, z' + (\alpha x^2 + \beta x + \gamma)\, z = 0.$$

If we endeavour to satisfy this by $z = e^{\frac{1}{2}px^2 + qx}$, we find that

$$e^{\frac{1}{2}px^2 + qx} \left[(px + q)^2 + p + (ax + b).(px + q) + \alpha x^2 + \beta x + \gamma \right]$$

must vanish identically.

That this may be so the following three equations must be satisfied:

$$\left.\begin{aligned} p^2 + ap + \alpha &= 0 \\ 2pq + aq + bp + \beta &= 0 \\ q^2 + p + bq + \gamma &= 0 \end{aligned}\right\}.$$

That these equations are, in general, inconsistent is clear.

I wish therefore to point out the error in Schlesinger's demonstration, which proceeds as follows.

Since the given equation has $x = \infty$ for its only singularity, any integral may be expanded in a power series in x, converging for all finite values of x.

Thus, if $v = \dfrac{1}{z}\dfrac{dz}{dx}$, v can also be expanded in the neighbourhood of $x = 0$ in a converging power series in x, the constant term in z being supposed other than zero.

CONTENTS.

	PAGE
A special continuant evaluated by Cayley (continued). By THOMAS MUIR	129
Quadratic partition tables.—Errata. By LT.-COL. CUNNINGHAM - -	132
Curves on a conicoid. By H. HILTON - - - - - - - -	136
Notes on the geometry of cubic surfaces.. By W. H. BLYTHE - -	139
On a paper-folding puzzle. By H. M. TAYLOR - - - - -	142
Note on a proposition stated by Schlesinger. By E CUNNINGHAM - -	144

Papers for insertion should be sent to Dr. Glaisher, Trinity College, Cambridge, or to Messrs. Metcalfe and Co. Limited, Printing Office, Trinity Street, Cambridge.

An index to vols. i—xxv has been published. It is issued gratis to subscribers, but may be obtained separately, price Two Shillings.

No. CCCCVI.] NEW SERIES. [February, 1905.

UNIVERSITY OF MICHIGAN LIBRARY 1905

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XXXIV.—NO. 10.

MACMILLAN AND CO.
London and Cambridge.
1905.

Price—One Shilling.

METCALFE AND CO. LIMITED, CAMBRIDGE.

But Schlesinger does not notice that the zeros of z are poles of v, so that the expansion of v only holds in general for a finite circle about $x=0$; and it is this which invalidates his conclusion.

For, as he shows, the analytic function v satisfies an equation which for $x=\infty$ reduces to the algebraic equation

$$\frac{v^n}{x^{np}} + A_1 \frac{v^{n-1}}{x^{(n-1)p}} + \ldots + A_r = 0,$$

where A_r is the coefficient of x^{rp} in ϕ_{rp}.

Consequently $\frac{v}{x^p}$ converges for $x=\infty$ to a definite limit which is a root of the equation

$$\rho^n + A_1\rho^{n-1} + A_2\rho^{n-2} + \ldots + A_r = 0.$$

He proceeds to say, "v is developable in positive integral powers of x, and consequently $W\left(=\frac{v}{x^p}\right)$ is a rational integral function of $\frac{1}{x}$ of degree 'p' at most."

After what has been said above it is clear that this does not follow; for v is not so developable in the neighbourhood of $x=\infty$.

e.g. suppose $z=(1-x)\,e^{\frac{1}{2}x^2}$,

then $$v = x + \frac{d}{dx}\{\log(1-x)\} = x - \frac{1}{1-x},$$

which for $|x|<1$ is developable about $x=0$. But it does not follow that since $\frac{v}{x}$ converges to unity for $x=\infty$, that $\frac{v}{x}$ is a linear function of $\frac{1}{x}$.

In the following section of his book, Schlesinger uses this proposition to demonstrate the existence of the normal series satisfying linear differential equations with rational coefficients, which Fabry investigated directly.

The failure of his fundamental result unfortunately destroys the simplicity of the method which relies on it. For in order to apply this method, one would first have to apply to the particular equation from which we started, the method of Fabry which is equally applicable directly to the equations in view.

ON CERTAIN CONDITIONALLY CONVERGENT MULTIPLE SERIES CONNECTED WITH THE ELLIPTIC FUNCTIONS.

By *G. H. Hardy*, Trinity College, Cambridge.

§ 1. In two earlier papers* I proved the convergence of a general class of multiple series of the form

$$\Sigma a_{n_1, n_2, \ldots, n_p} u_{n_1, n_2, \ldots, n_p},$$

of which the most interesting special cases were those obtained by taking

$$u_{n_1, n_2, \ldots, n_p} = \exp.\ (\sum_{\nu=1}^{p} \theta_\nu n_\nu),$$

and

$$a_{n_1, n_2, \ldots, n_p} = \phi\ (\sum_{\nu=1}^{p} a_\nu n_\nu),$$

$\phi(u)$ being a function of u which has zero for its limit for $u = \infty$ and continuous derivates $\phi'(u)$, $\phi^{(2)}(u)$, ..., $\phi^{(p)}(u)$ such that $\phi'(u) < 0$, $\phi''(u) > 0$, $\phi'''(u) < 0$, ...; and $a_1, a_2, \ldots, a_p$ being real and positive, and $\theta_1, \theta_2, \ldots, \theta_p$ real and no multiples of 2π. The limits of summation for each variable are 1 and ∞. In particular, I proved that

$$(1) \quad \Sigma \frac{\exp.\ (i \Sigma \theta_\nu n_\nu)}{(u + \Sigma a_\nu n_\nu)^\beta}$$

is convergent if u is real and positive and $\beta > 0$, and the summation is effected according to any *type*†

$$(1, 2, \ldots, \lambda)\,(\lambda + 1, \ldots, \mu) \ldots (\tau + 1, \ldots, p).$$

In a note attached to the first paper above mentioned, I indicated that the restriction that β, u, a_ν should be real was unnecessary.

It is in fact possible to prove that if (i) the values of u, $a_1, \ldots, a_p$, real or complex, are such that we can assign a constant H, for which

$$| u + \Sigma a_\nu n_\nu | > H,$$

---

* v. *Proc. Lond. Math. Soc.*, New Series, Vol. I., p. 124, and Vol. II, p. 190.

† For an explanation of this notation see a paper by Prof. Bromwich and myself, *Proc. Lond. Math. Soc.* (2), v. 2, p. 161.

and (ii) that value of the denominator of the general term is chosen, for which

$$(u + \Sigma a_\nu n_\nu)^\beta = \exp.\ \{\beta \log (u + \Sigma a_\nu n_\nu)\}$$

where $$-\pi \leqq R\left[\frac{1}{i}\log(u + \Sigma a^\nu n_\nu)\right] < \pi,$$

then the series (1) is convergent.

I do not now propose to give a detailed proof of this, which must, from the nature of the case, be less concise than that which suffices in the case when β, u, a_ν are all real and positive. The nature of the arguments which must be employed will be made sufficiently clear by the discussion of a particular case which follows. This is a double series first studied by Kronecker and employed by him to obtain very beautiful results in the theory of elliptic functions.* Both Kronecker and Prof. Lerch, who has also considered the series, wrote about it before the appearance of Pringsheim's memoirs on the general theory of conditionally convergent series; and their discussions of its convergence are therefore much less simple than one based on Pringsheim's theory, and the generalisation of Abel's lemma proved in my paper quoted above. My object in this paper is to consider its convergence from the latter point of view. I have also summed the series by a method practically identical with Kronecker's, but so as obtain the sum directly in terms of the ordinary Weierstrassian elliptic functions.

§ 2. The series in question is

$$(2) \quad \Sigma \frac{e^{(2m\theta+2n\phi)\pi i}}{a + m\omega_2 + n\omega_1},$$

where the limits of summation are $-\infty$ and $+\infty$ for each variable, θ and ϕ are real, positive, and less than unity, ω_1 and ω_2 are any two complex quantities such that the imaginary part of $\tau = \omega_2/\omega_1$ is positive, and a is any quantity such that $a + m\omega_2 + n\omega_1$ does not vanish for any values of m and n. The series is obviously not absolutely convergent. I shall prove first that *the series* (2) *is a convergent double series in Pringsheim's sense.*

Although the series (2) is not precisely of the type (1), ω_2 and ω_1 being complex, and the limits different, the proof

* *vide* Kronecker, *Sitzungsberichte d. k. P. Ak.* (1890 and neighbouring years, *passim*, and especially 1890 (2), pp. 123 *et seq*); also Lerch, *Rozprary Ceské Ak.*, 1895 and neighbouring years, *passim* (unfortunately written in Bohemian).

of its convergence is very similar to that which I gave of the convergence of (1). I shall therefore present the proof in as short a form as possible. Let

$$u_{m,n} = e^{(2m\theta+2n\phi)\pi i},$$

$$v_{m,n} = \frac{1}{a + m\omega_2 + n\omega_1}.$$

Then

$$(3)\quad \left(\sum_{0}^{M_1}\sum_{0}^{N_1} - \sum_{0}^{M}\sum_{0}^{N}\right) u_{m,n}v_{m,n}$$

$$= \left(\sum_{0}^{M}\sum_{N+1}^{N_1} + \sum_{M+1}^{M_1}\sum_{0}^{N} + \sum_{M+1}^{M_1}\sum_{N+1}^{N}\right) u_{m,n}v_{m,n}.$$

Now, by the extension of Abel's lemma proved in the paper referred to,

$$\sum_{M+1}^{M_1}\sum_{N+1}^{N_1} u_{m,n}v_{m,n} = \sum_{M+1}^{M_1-1}\sum_{N+1}^{N_1-1}\Delta_{m,n}\sum_{p=M+1}^{m}\sum_{q=N+1}^{n} u_{p,q}$$

$$+ \sum_{M+1}^{M_1-1}\Delta_m\sum_{M+1}^{m}\sum_{N+1}^{N_1} u_{p,q} + \sum_{N+1}^{N_1-1}\Delta_n\sum_{M+1}^{M_1}\sum_{N+1}^{n} u_{p,q} + \Delta\sum_{M+1}^{M_1}\sum_{N+1}^{N_1} u_{p,q},$$

where $$\Delta = v_{M_1,N_1},\quad \Delta_m = v_{m,N_1} - v_{m+1,N_1},$$

$$\Delta_n = v_{M_1,n} - v_{M_1,n+1},$$

and $$\Delta_{m,n} = v_{m,n} - v_{m+1,n} - v_{m,n+1} + v_{m+1,n+1}.$$

The modulus of this is less than a constant multiple of

$$(4)\quad \sum_{M+1}^{M_1}\sum_{N+1}^{N_1}|\Delta_{m,n}| + \sum_{M+1}^{M_1-1}|\Delta_m| + \sum_{N+1}^{N_1-1}|\Delta_n| + |\Delta|,$$

since there is a constant K such that

$$\left|\sum_{m_1}^{m_2}\sum_{n_1}^{n_2} u_{p,q}\right| < K$$

for *all* values of m_1, n_1, m_2, n_2.

Now it is easy to see that

$$\Delta_{m,n} = \frac{\omega_1\omega_2\{a + (2m+1)\omega_2 + (2n+1)\omega_1\}}{(m,n)(m+1,n)(m,n+1)(m+1,n+1)},$$

where $$(m,n) = a + m\omega_2 + n\omega_1,$$

and therefore that, unless $m = n = 0$,

$$|\Delta_{m,n}| < \frac{K}{|m\omega_2 + n\omega_1|^3},$$

where K is independent of m, n, and a. But the right-hand side is the general term of a series known, by Eisenstein's theorem, to be absolutely convergent. It follows that given σ we can choose M_0, N_0, so that

$$\sum_{M+1}^{M_1} \sum_{N+1}^{N_1} |\Delta_{m,n}| < \sigma,$$

for all values of M, M_1, N, N_1, such that $M_1 > M \geqq M_0$, $N_1 > N \geqq N_0$.

Similar arguments may be applied to each of the other terms on the right of (4).

It follows that given σ we can choose M_0 and N_0, so that

$$\left| \sum_{M+1}^{M_1} \sum_{N+1}^{N_1} u_{m,n} v_{m,n} \right| < \sigma,$$

for all values of M, M_1, N, N_1, such that $M_1 > M \geqq M_0$, $N_1 > N \geqq N_0$, and moreover that M_0 and N_0 are independent of a.

A similar line of argument may be applied to the two other terms which occur on the right-hand side of (3). We conclude then that the double series

$$\sum_{0}^{\infty} \sum_{0}^{\infty} u_{m,n} v_{m,n}$$

is convergent. Exactly the same process proves that

$$\sum_{0}^{\infty} \sum_{-\infty}^{-1}, \quad \sum_{-\infty}^{-1} \sum_{0}^{\infty}, \quad \sum_{-\infty}^{-1} \sum_{\infty}^{-1}$$

are convergent. Therefore (2) is convergent, when summed as a *double* series; that is to say

$$\sum_{-M'}^{M} \sum_{-N'}^{N} \frac{e^{(2m\theta + 2n\phi)\pi i}}{a + m\omega_2 + n\omega_1}$$

tends to a finite limit when M, N, M', N' are made to tend simultaneously and independently to ∞; and moreover, that this convergence is *uniform* for all values of a which lie in any continuous domain which does not include any point, such that

$$a + m\omega_2 + n\omega_1 = 0,$$

for any values of m and n.

It follows by a theorem of Pringsheim's that *if* the series is convergent when summed as a repeated series, either by rows or by columns, its sum according to that method of summation is equal to its sum as a double series. Propositions substantially equivalent to these are proved by Kronecker with the aid of certain definite integrals. That the series is so convergent as a repeated series is easily proved, either by general considerations or by direct summation. The summation which follows is, as I stated above, practically the same as Kronecker's except in notation.

Summation of the series.

§ 3. If $$w = -\frac{a+m\omega}{\omega_1},$$

$$\sum_{n=-\infty}^{\infty} \frac{e^{2n\phi\pi i}}{a+m\omega_2+n\omega_1} = -\frac{1}{\omega_1}\sum_{-\infty}^{\infty}\frac{e^{2n\phi\pi i}}{w-n} = -\frac{2\pi i}{\omega_1}\frac{e^{2w\phi\pi i}}{e^{2\omega\pi i}-1}.*$$

$$= -\frac{2\pi i}{\omega_1}\frac{\exp.\left\{-2\pi i\phi\left(\frac{a+m\omega_2}{\omega_1}\right)\right\}}{\exp.\left\{-2\pi i\left(\frac{a+m\omega_2}{\omega_1}\right)\right\}-1}.$$

First suppose $m=\mu>0$. Then

$$\sum_{n=-\infty}^{\infty}\frac{e^{(2m\theta+2n\phi)\pi i}}{a+m\omega_2+n\omega_1} = -\frac{2\pi i}{\omega_1}\frac{q^\mu z}{1-q^{2\mu}z^2}t^{2\mu}\exp.\left\{(1-2\phi)\frac{\pi i a}{\omega_1}\right\},$$

where $$q=e^{\pi i\tau},\quad z=e^{\frac{\pi i a}{\omega_1}},$$

$$t=\exp.\left[\frac{\pi i}{\omega_1}\{\omega_1\theta-\omega_2(\phi-\tfrac{1}{2})\}\right].$$

On the other hand, if m is negative, and $-m=\mu>0$,

$$\sum_{n=-\infty}^{\infty}\frac{e^{(2m\theta+2n\phi)\pi i}}{a+m\omega_2+n\omega_1} = \frac{2\pi i}{\omega_1}\frac{q^\mu z^{-1}}{1-q^{2\mu}z^{-2}}t^{-2\mu}\exp.\left\{(1-2\phi)\frac{\pi i a}{\omega_1}\right\}.$$

Thus the terms of (2) for which $m \gtrless 0$ contribute

$$-\frac{2\pi i}{\omega_1}\exp.\left\{(1-2\phi)\frac{\pi i a}{\omega_1}\right\}\sum_{\mu=1}^{\infty}\left[\frac{q^\mu t^{2\mu}z}{1-q^{2\mu}z^2}-\frac{q^\mu t^{-2\mu}z^{-1}}{1-q^{-2\mu}z^{-2}}\right].$$

* Kronecker, *Vorlesungen ueber Integrale*, p. 105.

Again, the terms for which $m = 0$ contribute

$$\sum_{-\infty}^{\infty} \frac{e^{2n\phi\pi i}}{a + n\omega_1} = -\frac{2\pi i}{\omega_1} \frac{\exp.\left(-\frac{2\pi i\phi a}{\omega_1}\right)}{\exp.\left(-\frac{2\pi i a}{\omega_1}\right) - 1}$$

$$= \frac{2\pi i}{\omega_1} \exp.\left\{(1-2\phi)\frac{\pi i a}{\omega_1}\right\} \frac{1}{z - z^{-1}}.$$

Thus in all we obtain

$$\frac{2\pi i}{\omega_1} \exp.\left\{(1-2\phi)\frac{\pi i a}{\omega_1}\right\} \left[\frac{1}{z-z^{-1}} - \sum_1^{\infty}\left(\frac{q^{\mu}t^{2\mu}z}{1-q^{2\mu}z^2} - \frac{q^{\mu}t^{-2\mu}z^{-1}}{1-q^{2\mu}z^{-2}}\right)\right].$$

But*

$$\frac{\sigma_2(u+v)}{\sigma_2(v)\,\sigma(u)} e^{-\frac{\eta_1 uv}{\omega_1}} = \frac{2\pi i}{\omega_1}\left[\frac{1}{z-z^{-1}} - \sum_1^{\infty}\left(\frac{q^{\mu}t^{2\mu}z}{1-q^{2\mu}z^2} - \frac{q^{\mu}t^{-2\mu}z^{-1}}{1-q^{2\mu}z^{-2}}\right)\right],$$

where† $$z = e^{\frac{i\pi u}{\omega_1}}, \quad t = e^{\frac{i\pi v}{\omega_1}}.$$

Thus the sum of our series is

$$(5) \quad \exp.\left\{(1-2\phi)\frac{\pi i a}{\omega_1}\right\} \frac{\sigma_2(v)\,\sigma(u)}{\sigma_2(u+v)} e^{-\frac{\eta_1 uv}{\omega_1}},$$

where $$u = a, \quad v = \omega_1\theta - \omega_2(\phi - \tfrac{1}{2}).$$

Now $$\sigma_2(v) = \frac{\sigma(u + \frac{1}{2}\omega_2)}{\sigma(\frac{1}{2}\omega_2)} e^{-\frac{1}{2}\eta_2 u},$$

so that (5) reduces to

$$e^{a(\eta_2\phi - \eta_1\theta)} \frac{\sigma(a + \omega_1\theta - \omega_2\phi)}{\sigma(a)\,\sigma(\omega_1\theta - \omega_2\phi)}.$$

Thus we arrive at the result

$$(6) \quad \Sigma \frac{e^{(2m\theta + 2n\phi)\pi i}}{a + m\omega_2 + n\omega_1} = e^{a(\eta_2\phi - \eta_1\theta)} \frac{\sigma(a + \omega_1\theta - \omega_2\phi)}{\sigma(a)\,\sigma(\omega_1\theta - \omega_2\phi)},$$

the periods of the elliptic functions being ω_1, ω_2.

§ 4. From this result a variety of other interesting formulæ may be deduced. In the first place we observe that the series

* Halphen, *Fonctions Elliptiques*, i., p. 422.

† I write ω_1, ω_2 for Halphen's $2\omega_1$, $2\omega_2$.

is uniformly convergent for values of a lying in any domain which does not include any of the points $-m\omega_2 - n\omega_1$. Now it is easy to see that if we remove the term for which $m=n=0$, the resultant series is uniformly convergent for a region of values of a including $a=0$. Hence

$$\Sigma' \frac{e^{(2m\theta+2n\phi)\pi i}}{m\omega_2+n\omega_1} = \lim_{a=0}\left[-\frac{1}{a}+e^{(\eta_2\phi-\eta_1\theta)a}\frac{\sigma(a+\omega_1\theta-\omega_2\phi)}{\sigma(a)\,\sigma(\omega_1\theta-\omega_2\phi)}\right]$$
$$= \zeta(\omega_1\theta-\omega_2\phi)-\eta_1\theta+\eta_2\phi,$$

the dash over the sign of summation implying that the pair of values $m=n=0$ are excluded.

§ 5. Again it is easy to justify differentiating (6) any number of times with respect to a. In fact it is to be observed that the general theorems:

(i) A series is continuous for values of a parameter within a certain domain, if uniformly convergent throughout that domain;

(ii) The series may be integrated term by term along any path within the domain: and

(iii) The corresponding theorems for differentiation

are just as true for (proper) double series as for simple series; and the uniform convergence of the various derived series may be proved in this case just as in § 3.

Differentiate (6) once, subtract the term $-\frac{1}{a^2}$, and proceed to the limit for $a=0$. We obtain

$$(8)\quad \Sigma' \frac{e^{(2m\theta+2n\phi)\pi i}}{(m\omega_2+n\omega_1)^2} = \tfrac{1}{2}\{p(u)-\zeta^2(u)\}+v\zeta(u)-\tfrac{1}{2}v^2,$$

where $$u=\omega_1\theta-\omega_2\phi,\quad v=\eta_1\theta-\eta_2\phi.$$

Again, we may differentiate (6) twice, obtaining

$$\Sigma \frac{e^{(2m\theta+2n\phi)\pi i}}{(a+m\omega_2+n\omega_1)^2} = \tfrac{1}{2}\left(\frac{\partial}{\partial a}\right)^2\left\{e^{a(\eta_2\phi-\eta_1\theta)}\frac{\sigma(a+\omega_1\theta-\omega_2\phi)}{\sigma(a)\,\sigma(\omega_1\theta-\omega_2\phi)}\right\}.$$

The series on the left is absolutely and uniformly convergent for all real values of θ and ϕ. Making $\theta=\phi=0$, we obtain

$$\Sigma \frac{1}{(a+m\omega_2+n\omega_1)^2} = \tfrac{1}{2}\left\{\frac{\sigma'''(a)}{\sigma(a)}-\frac{3\sigma'(a)\,\sigma''(a)}{\sigma^2(a)}+\frac{2\sigma'^3(a)}{\sigma^3(a)}\right\}$$
$$= -\tfrac{1}{2}p'(a),$$

which agrees with the definition from which the theory of the Weierstrassian functions usually starts.

§ 6. The series

$$\Sigma' \frac{e^{(2m\theta+2n\phi)\pi i}}{am^2+2bmn+cn^2} \quad (ac-b^2>0)$$

and other series considered by Kronecker and M. Lerch, may be treated in the same manner.

ON THE FACTORS OF FERMAT'S NUMBERS.

By *Thorold Gosset.*

NUMBERS of the form $2^{2^n}+1$ were believed by Fermat to be primes. This is true when n is less than 5. When, however, n is equal to, or greater than, 5, the numbers are not necessarily prime, and several prime factors of these numbers have been discovered. As all such factors are primes of the form $4m+1$, each must be in one way only the sum of two squares.

In fact every real factor (other than the number itself when n is less than 4) must be capable of being written in one of the four following forms:—

$$(16p\pm4)^2 + (64q\pm25)^2 \quad \text{.....(i.)},$$

$$(32p\pm8)^2 + (256q\pm31)^2 \quad \text{.....(ii.)},$$

$$(64p\pm16)^2 + (256q\pm127)^2 \quad \text{.....(iii)},$$

$$(32p)^2 + (256q\pm1)^2 \quad \text{.....(iv.)}.$$

A real prime factor of one of Fermat's numbers can consequently be split into two complex prime factors. One of these will be a factor $2^{2^{n-1}}+\sqrt{(-1)}$, and the other a factor of $2^{2^{n-1}}-\sqrt{(-1)}$.

In the following table I have given a list of the known prime factors of Fermat's numbers with their resolutions into their complex constituents. The complex factor with the

upper sign is in each case a factor of $2^{2^{n-1}} + \sqrt{(-1)}$, and the factor with the lower sign is a factor of $2^{2^{n-1}} - \sqrt{(-1)}$.

I have also verified that all the numbers given below as real factors do in fact divide without a remainder the corresponding Fermat's numbers.

For the complete list of real factors discovered up to the present time I am indebted to Mr. W. W. Rouse Ball.

Fermat's number.	Real Prime Factor.	Form	Complex Prime Factors.
$2^{2^5}+1$	641	i	$25 \mp 4\sqrt{(-1)}$
,,	6 700 417	i	$2\,556 \pm 409\sqrt{(-1)}$
$2^{2^6}+1$	274 177	i	$516 \mp 89\sqrt{(-1)}$
,,	67 280 421 310 721	i	$8\,083\,111 \pm 1\,394\,180\sqrt{(-1)}$
$2^{2^9}+1$	2 424 833	iii	$1\,552 \mp 127\sqrt{(-1)}$
$2^{2^{11}}+1$	319 489	i	$540 \mp 167\sqrt{(-1)}$
,,	974 849	i	$985 \pm 68\sqrt{(-1)}$
$2^{2^{12}}+1$	114 689	i	$260 \mp 217\sqrt{(-1)}$
,,	26 017 793	iv	$4\,672 \mp 2\,047\sqrt{(-1)}$
,,	63 766 529	iii	$7\,295 \pm 3\,248\sqrt{(-1)}$
$2^{2^{16}}+1$	13 631 489	i	$3\,692 \mp 25\sqrt{(-1)}$
$2^{2^{23}}+1$	167 772 161	i	$12\,455 \pm 3\,556\sqrt{(-1)}$
$2^{2^{36}}+1$	2 748 779 069 441	ii	$1\,640\,929 \pm 236\,920\sqrt{(-1)}$
$2^{2^{38}}+1$	6 597 069 766 657	ii	$2\,049\,336 \mp 1\,548\,319\sqrt{(-1)}$

ON THE ASYMPTOTIC EXPANSION OF AN ANALYTIC FUNCTION.

By *E. Cunningham*, St. John's College, Cambridge.

THE function $\phi(x)$ is said to be asymptotically represented by the series

$$a_0 + a_1 x + \ldots + a_n x^n + \ldots$$

within a given angle S at the point $x = 0$, if a quantity ρ exists, such that at all points within both S and the circle $|x| = \rho$, ρ being sufficiently small,

$$\left| \frac{\phi(x) - a_0 + a_1 x + \ldots + a_n x^n}{x^n} \right| < \epsilon,$$

ϵ being any assignable small quantity.

It is well known that if two analytic functions possess asymptotic expansions within a common region, then the sum or product of those functions is represented asymptotically within the same region by the formal sum or product of those expansions. But I believe that it has not been hitherto observed that the quotient of the functions is also asymptotically represented within the same region by the formal expansion of the quotient of the series, except in the case when a_0 is zero. This fact is proved below.

It will be sufficient to prove that if, within a given region S, the function $\phi(x)$ is asymptotically represented by the series

$$a_0 + \ldots + a_n x + \ldots,$$

and if, by a formal expansion of the expression

$$\frac{1}{a_0 + \ldots + a_n x^n + \ldots},$$

the series $b_0 + b_1 x + \ldots + b_n x^n \ldots$

be obtained, then within the region S the function $\frac{1}{\phi(x)}$ is asymptotically represented by the last-written series.

Within the region σ, common to S and the interior of the circle $|x| = \rho$,

$$\phi(x) = a_0 + a_1 x + \ldots + a_n x^n + \epsilon_n x^n,$$

where ϵ_n is a function of x less in absolute value than the small constant ϵ.

Thus within σ

$$\phi(x) = \frac{1}{a_0 + a_1 x + \ldots + (a_n + \epsilon_n) x^n}$$

$$= \frac{1}{a_0 + \ldots + a_n x^n} - \frac{\epsilon_n}{(a_0 + \ldots + a_n x^n)\{a_0 + \ldots + (a_n + \epsilon_n) x^n\}} \cdot x^n.$$

Now a_0 being a finite quantity other than zero, the roots of the equation

$$a_0 + a_1 x + \ldots + a_n x^n = 0$$

all lie outside a certain finite circle.

Further, since within the region σ

$$|\epsilon_n| < \epsilon,$$

ϵ being an arbitrary small quantity, there exists a certain finite circle about $x=0$, such that within the region common to it and to σ

$$a_0 + \ldots + (a_n + \epsilon_n) x^n$$

does not vanish, but has a finite lower limit.

Consequently there exists a finite quantity ρ_1 such that within the region S and for $|x| > \rho$

$$|(a_0 + \ldots + a_n x^n)\{a_0 + \ldots + (a_n + \epsilon_n) x^n\}| > A,$$

A being a finite quantity.

Consequently within the same region

$$\left| \frac{\epsilon_n}{(a_0 + \ldots + a_n x^n)\{a_0 + \ldots + (a_n + \epsilon_n) x^n\}} \right| < A\epsilon,$$

and therefore

$$\left| \frac{1}{\phi(x)} - \frac{1}{a_0 + \ldots + a_n x^n} \right| < A\epsilon |x|^n.$$

Now, since within this same region the polynomial $(a_0 + \ldots + a_n x^n)$ does not vanish, the function $\frac{1}{a_0 + \ldots + a_n x^n}$ can be expanded in a convergent series of ascending integral powers of x, and this series will certainly as far as the term involving x^n be identical with the series

$$b_0 + b_1 x + \ldots + b_n x^n + \ldots.$$

Hence

$$\frac{1}{a_0 + \ldots + a_n x^n} - (b_0 + b_1 x + \ldots + b_n x^n) = x^{n+1}\{c_{n+1} + c_{n+2} x + \ldots\},$$

where within the region considered the series

$$c_{n+1} + c_{n+2} x + \ldots$$

converges.

Thus

$$\left| \frac{1}{a_0 + a_1 x + \ldots + a_n x^n} - (b_0 + b_1 x + \ldots + b_n x^n) \right| < |x^{n+1}| C,$$

where C is a finite constant, and for $|x| < \eta$, this is less than $C\eta |x|^n$.

Thus

$$\left| \frac{1}{\phi(x)} - (b_0 + b_1 x + \ldots + b_n x^n) \right| < (A\epsilon + B\eta) |x|^n,$$

and ϵ and η can be taken sufficiently small to make $(A\epsilon + B\eta)$ less than any assigned quantity ϵ.

If follows that $\frac{1}{\phi(x)}$ is asymptotically represented by the series $b_0 + b_1 x + \ldots + b_n x^n + \ldots$ within the region S, provided a_0 is not zero.

It follows that, if the functions $f, \phi, \psi, \ldots$ possess asymptotic expansions within a common region, not only is any rational *integral* function of f, ϕ, ψ asymptotically represented within the same region by the formal development in series of the same function of the several expansions, but that the same is true of *any rational function whatever* of $f, \phi, \psi, \ldots$, provided that there is no factor in the denominator which vanishes for $x = 0$.

AN EXTENSION OF THE THEORY OF GROUPS OF MOVEMENTS.

By *Harold Hilton.*

THE "movements" usually considered in the above theory* are those consisting of a reflexion in a plane, successive reflexions in two planes (rotation), successive reflexions in three planes (rotatory-reflexion), all of which leave the distance between any two points unaltered. We shall discuss a class of "movements" which have not this property; and to simplify our language shall extend our definition of reflection, &c.

Let j be the real or imaginary intersection of a real plane t with a real conicoid. Let A be any point in t, and let its polar with respect to j meet j in H, K. Let r be any plane through HK, and let any straight line $APFP'$ through A meet r in F, so that the range (AF, PP') is harmonic: then

* See C. Jordan, *Ann. di matematica pura ed applicata*, II., 2; A. Schoenflies, *Krystallsysteme und Krystallstructur*; H. Hilton, *Math. Crystallography*, &c.

P, P' will be called "reflexions of each other in r with respect to j." Any straight line through A will be called "perpendicular" to r.

Let l be any straight line through A. If l lies in t, successive reflexions in any number of planes through l move any point along a straight line through the polar of l with respect to j. The resultant of two such reflexions will be called a "translation." If p, q, y, z are four real planes through l such that successive reflexions in p and q are equivalent to successive reflexions in y and z, the segment intercepted by p and q on any straight line parallel to t is equal to the segment intercepted by y and z.

Now suppose l does not lie in t, and let p and q be two real planes through l. Let h, k be the planes through l touching j at H, K, and let the cross-ratio of (hk, pq) be $e^{i\alpha}$. Let P be brought to P' by successive reflexions in p and q; then P, P' lie on a conic touching h, k at H, K, and the cross-ratio of $l(HK, PP')$ is $e^{2i\alpha}$. The two successive reflexions are said to be equivalent to a "rotation through α about l." Three cases arise: (1) h and k are an imaginary pair, α is real; (2) h and k are real and (hk, pq) is a non-overlapping involution, α is of the form $2n\pi + i\beta$ (β real and n integral or zero); (3) h and k are real and (hk, pq) is an overlapping involution, α is of the form $(2n+1)\pi + i\beta$. We shall take the real part of $\alpha < 2\pi$ and $\not< 0$; this evidently involves no loss of generality. In case (1) the rotation is an operation with a real period; this is not so in cases (2) and (3). In (1) and (2) P and P' are on the same side of HK, and in (3) on opposite sides. If in (1) or (3) $\alpha = \pi$, PP' meets l; in this case we call the rotation a "reversal" (umklappung, umwendung).

If l meets j at A, α must be zero to a first approximation. In this case the rotation may be measured by the length intercepted by p and q on any fixed straight line u parallel to the tangent at A to j; for if p, q, y, z are four planes through l, such that successive reflexions in p and q are equivalent to successive reflexions in y and z, p and q intercept on u a segment equal to that intercepted by y and z.

Two successive rotations through α and β about lines l and m through a point O are, in general, equivalent to a rotation through γ about an axis n through O. If l, m, n lie in a plane, this plane touches j and $\gamma = \alpha + \beta$. The only other case in which $\gamma = \alpha + \beta$ is that in which O lies in t.

If the straight line $POP'F$, passing through a fixed point O, meets t in F, so that (PP', OF) is harmonic, P is said to be brought to P' by an "inversion" about O. The inversion is

equivalent to a reversal about any line l through O and a reflexion in the plane through O perpendicular to l.

The "movement" consisting of a rotation through α about any line l and reflexion in a perpendicular plane is called a "rotatory-reflexion" through α about l. A rotatory-reflexion through π is equivalent to an inversion.

As an illustration we shall enumerate the groups of movements, leaving one point O (not lying in t) unmoved, containing no infinitesimal operation, and involving only a finite number of rotation-axes other than reversal-axes. Such a group is finite if α is real for every rotation-axis, otherwise infinite.

A. j is imaginary.

These groups may be at once inferred from the ordinary symmetry-groups of crystallography*; for j may be transformed into the imaginary circle at infinity by a real linear transformation.

B. j is real.

I. We have groups with a rotation or rotatory-reflexion through α about an axis l through O, where $n\alpha = 2\pi$, n being a finite positive integer. Denote by R the rotation and by R' the rotatory-reflexion through α about l, by U the reversal about l, by V the reversal about a line through O perpendicular to l, by S the reflexion in a plane through O perpendicular to l, by S' the reflexion in a plane passing through l but through no other axis, by I the inversion about O.

There are finite groups obtained by combining powers and products of (1) R; (2) R'; (3) R, S; (4) R, S'; (5) R, V; (6) R, V, S; (7) R, V, S'. These groups contain respectively

$$n,\ \tfrac{1}{2}n[3-(-1)^n],\ 2n,\ 2n,\ 2n,\ 4n,\ 4n$$

movements. (4), (6), (7) involve reflexions in n symmetry-planes through l; (5), (6), (7) involve reversals in n axes perpendicular to l. In (6) the symmetry-planes contain the reversal axes; in (7) they bisect the angles between adjacent reversal-axes.

II. We have groups with a rotation or rotatory-reflexion through α about one axis l through O, where α is a pure imaginary.

There are infinite groups derived from (1) R; (2) R, U; (3) R'; (4) R', U; (5) R, S; (6) R, S, U; (7) R, S'; (8) R, S', U; (9) R, I; (10) R, V; (11) R, V, U; (12) R, V, S; (13) R, V, S, U; (14) R, V, I; (15) R', V; (16) R', V, U.

* The restriction of α to π, $\frac{1}{3}(2\pi)$, $\frac{1}{2}\pi$, $\frac{1}{4}\pi$ is, of course, removed.

III. We have infinite groups with a rotation or rotatory-reflexion through α about one axis, where α is of the form $\frac{1}{2}\pi + i\beta$, β being a real quantity $\neq 0$. These groups exactly correspond to the 9 groups of II. not involving U. No *distinct* groups of III. correspond to the 7 groups of II. containing U.

IV. We have groups with rotations about a line l through O meeting j.

There are infinite groups derived from (1) R; (2) R, S'; (3) R, I; (4) R, V; (5) R, V, I. The reversal-axes of (4) and (5) are infinite in number and lie in a plane through l touching j.

We have altogether, in § A, 14, and, in § B, $7+16+9+5=37$ distinct groups.

ON A PERFECT PLANE SET.

By *W. H. Young, Sc.D.*

If we attempt, as, for instance, was done by Veltmann, to construct perfect sets, dense nowhere, in the plane by a method analogous to that originally introduced by H. J. S. Smith for the straight line, the sets obtained necessarily contain straight lines or curves. Examples of sets which contain no straight lines or curves, using this latter word in its most general sense, possess therefore a special interest. I propose here to give an example of such a set. It has the remarkable property that its projection on each of the coordinate axes is the whole of the continuum in a certain segment.

Take a square, choose one of its diagonals, divide it into three parts, and construct squares with these three parts as diagonals. Mark the corners of the middle square with black dots, these points are to belong to the set. In each of the small squares repeat this construction, choosing, in the case of the extreme squares, the same diagonal as before, and, in the case of the middle square, the other diagonal. Repeat this construction *ad infinitum*. The required set is composed of the black dots constructed at all stages and their limiting points.

It may be proved that a Jordan curve can be drawn through all the points of this set, but in such a way that the set is dense nowhere on the Jordan curve.

I content myself here with these indications, as I hope to give a more detailed account of this set, with proofs of its properties, elsewhere.

January 19th, 1905.

CONTENTS.

	PAGE
Note on a proposition stated by Schlesinger (continued). By E. CUNNINGHAM	145
On certain conditionally convergent multiple series connected with the elliptic functions. By G. H. HARDY	146
On the factors of Fermat's numbers. By THOROLD GOSSET	153
On the asymptotic expansion of an analytic function. By E. CUNNINGHAM	155
An extension of the theory of groups of movements. By HAROLD HILTON	157
On a perfect plane set. By W. H. YOUNG	180

Papers for insertion should be sent to Dr. Glaisher, Trinity College, Cambridge, or to Messrs. Metcalfe and Co. Limited, Printing Office, Trinity Street, Cambridge.

An index to vols. i—xxv has been published. It is issued gratis to subscribers, but may be obtained separately, price Two Shillings.

No. CCCCVII.] NEW SERIES. [March, 1905.

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XXXIV.—NO. 11.

MACMILLAN AND CO.
London and Cambridge.
1905.

Price—One Shilling.

METCALFE AND CO. LIMITED, CAMBRIDGE.

CONTENTS.

PAGE

Note on a proposition stated by Schlesinger (continued). By E. CUNNINGHAM - - - - - - - - - - - - - - - 145

On certain conditionally convergent multiple series connected with the elliptic functions. By G. H. HARDY - - - - - - - - - 146

On the factors of Fermat's numbers. By THOROLD GOSSET - - - - 153

On the asymptotic expansion of an analytic function. By E. CUNNINGHAM 155

An extension of the theory of groups of movements. By HAROLD HILTON 157

On a perfect plane set. By W. H. YOUNG - - - - - - - - 180

Papers for insertion should be sent to Dr. Glaisher, Trinity College, Cambridge, or to Messrs. Metcalfe and Co. Limited, Printing Office, Trinity Street, Cambridge.

An index to vols. i—xxv has been published. It is issued gratis to subscribers, but may be obtained separately, price Two Shillings.

No. CCCCVII.] NEW SERIES. [March, 1905.

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XXXIV.—NO. 11.

MACMILLAN AND CO.
London and Cambridge.
1905.

Price—One Shilling.

METCALFE AND CO. LIMITED, CAMBRIDGE.

ON POLYGONS INSCRIBED IN A BINODAL QUARTIC AND CIRCUMSCRIBED ABOUT A CONIC.

By *R. A. Roberts.*

I PROPOSE to show here that an infinite number of polygons of an even number of sides can be inscribed in a binodal quartic, so that the sides touch a particular conic of a system having quartic contact with the curve. I consider the quartic in the form of the bicircular quartic, which, as is well known, can be generated as the envelope of a circle which has its centre on a given conic F and cuts orthogonally a given circle J.

Thus, putting

$$F=\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}-1=0 \quad \ldots\ldots\ldots(1),$$

$$J=x^2+y^2-k^2=0 \quad \ldots\ldots\ldots\ldots\ldots(2),$$

the quartic is

$$V=(x^2+y^2-2\alpha x-2\beta y+k^2)^2-4(a^2x^2+b^2y^2)=0 \ldots(3).$$

Now a conic having quartic contact with (3) is

$$U=\theta^2+\theta(x^2+y^2-2\alpha x-2\beta y+k^2)+a^2x^2+b^2y^2=0 \ldots(4),$$

and this belongs to the system that I propose to consider.

Putting

$$du=\frac{dx}{\frac{dV}{dy}}=\frac{ds}{\sqrt{\left\{\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2\right\}}}=\frac{rd\theta}{\frac{dV}{dr}} \quad \ldots\ldots(5),$$

I observe that I have shown in an article contributed to the *Proceedings of the London Mathematical Society* that if a line meet a quartic curve in the points A, B, C, D, then

$$du_1=\frac{OA}{AB.AC.AD},\quad du_2=\frac{OB}{BA.BC.BD},\quad du_3=\&c.\ldots(6),$$

where O is the point of contact of the line with its envelope.

Now, taking two of the conics (4), the quartic (3) can be written in the form

$$U_1U_2-[\theta_1\theta_2+\tfrac{1}{2}(\theta_1+\theta_2)(x^2+y^2-2\alpha x-2\beta y+k^2)+a^2x^2+b^2y^2]^2=0 \quad \ldots\ldots(7),$$

from which it follows that a line meets the quartic in four points given by $f(x)=(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$, and the conics U_1, U_2 in four points given by

$$\phi(x)=(x-x_1)(x-x_2)(x-x_3)(x-x_4),$$

so that
$$\Sigma\frac{\sqrt{\phi(\alpha)}}{f'(\alpha)}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(8),$$

where the summation refers to α, β, γ, δ; for (7) shows that $l\phi(x)+mf(x)=$ a perfect square, $=(\lambda+\mu x+\nu x^2)^2$, say, and we obtain (8) from this by the substitution of α, β, γ, δ successively and the elimination of λ, μ, ν. But the square of the line at infinity is one of the conics of the system (4) corresponding to $\theta=\infty$, so that $x_3=x_4=\infty$ in this case, and then, if we take a tangent of the other conic $\phi(x)=(x-x_1)^2$, (8) gives

$$\Sigma\pm\frac{\alpha-x_1}{f'(\alpha)}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(9),$$

but, since $\Sigma\frac{\alpha-x_1}{f'(\alpha)}=0$, identically, we get

$$\frac{\alpha-x_1}{f'(\alpha)}+\frac{\beta-x_1}{f'(\beta)}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(10),$$

and its equivalent

$$\frac{\gamma-x_1}{f'(\gamma)}+\frac{\delta-x_1}{f'(\delta)}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(11),$$

and two other similar pairs of equations. That is, with the notation of (6),

$$\frac{OA}{AB.AC.AD}+\frac{OB}{BA.BC.BD}=0,$$

which shows that $du_1+du_2=0$, whence

$$u_1+u_2=\sigma \quad \ldots\ldots\ldots\ldots\ldots\ldots(12),$$

where σ is a constant depending on the particular conic involved. Again, since for four collinear points

$$u_1+u_2+u_3+u_4=\omega \quad \ldots\ldots\ldots\ldots\ldots\ldots(13),$$

where ω is an absolute constant of the curve, we have

$$u_3+u_4=\omega-\sigma \quad \ldots\ldots\ldots\ldots\ldots\ldots(14).$$

We can now consider the problem of the inscription of the polygons. For a quadrilateral we can take

$$u_1 + u_2 = \sigma,\ u_2 + u_3 = \omega - \sigma,\ u_3 + u_4 = \sigma,\ u_4 + u_1 = \omega - \sigma \quad \ldots\ldots\ldots(15),$$

and these equations can coexist for a singly infinite number of values of u_1, u_2, u_3, u_4, provided

$$4\sigma = 2\omega \quad \ldots\ldots\ldots(16).$$

In the same way for a polygon of $2n$ sides, we obtain

$$2n\sigma = 2\omega \quad \ldots\ldots\ldots(17).$$

The actual number of conics of the system (4) determined by these equations will be investigated further on.

Expressing that the line joining the points x_1, y_1; x_2, y_2 touches the conic (4), we obtain, if

$$x^2 + y^2 - 2\alpha x - 2\beta y + k^2 = S,$$

$$(\theta^2 + \theta S_1 + a^2 x_1^2 + b^2 y_1^2)\ (\theta^2 + \theta S_2 + a^2 x_2^2 + b^2 y_2^2)$$

$$= [\theta^2 + \theta\{x_1 x_2 + y_1 y_2 - \alpha(x_1 + x_2) - \beta(y_1 + y_2) + k^2\} + a^2 x_1 x_2 + b^2 y_1 y_2]^2 \quad \ldots\ldots(18).$$

But if the points lie on the quartic $S^2 = 4(a^2 x^2 + b^2 y^2)$, we have $\theta^2 + \theta S + a^2 x^2 + b^2 y^2 = (\theta + \frac{1}{2}S)^2$. Hence, substituting in (18), and extracting the square root with the positive sign, we get

$$\theta^2 + \tfrac{1}{2}\theta(S_1 + S_2) + \tfrac{1}{4}S_1 S_2$$

$$= \theta^2 + \theta\{x_1 x_2 + y_1 y_2 - \alpha(x_1 + x_2) - \beta(y_1 + y_2) + k^2\} + a^2 x_1 x_2 + b^2 y_1 y_2,$$

or

$$2\sqrt{\{(a^2 x_1^2 + b^2 y_1^2)\ (a^2 x_2^2 + b^2 y_2^2)\}} + \theta\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}$$

$$= 2(a^2 x_1 x_2 + b^2 y_1 y_2) \ldots\ldots(19).$$

Transforming this to polar coordinates, and observing that from (3) we have

$$\rho + \frac{k^2}{\rho} = 2p, \quad \theta = \omega,$$

where p is the perpendicular from the origin on the tangent

to the conic F, and ω is the angular position of that perpendicular, we obtain

$$(p_1 - \alpha\cos\omega_1 - \beta\sin\omega_1)(p_2 - \alpha\cos\omega_2 - \beta\sin\omega_2) + \frac{\theta}{k^2}p_1p_2$$
$$= (a^2+\theta)\cos\omega_1\cos\omega_2 + (b^2+\theta)\sin\omega_1\sin\omega_2$$
$$+ \frac{\theta}{k^2}\sqrt{\{(p_1^2-k^2)(p_2^2-k^2)\}}\ldots(20),$$

where p_1, ω_1; p_2, ω_2 are the coordinates of the two tangents of the conic F, namely,

$$F = (p - \alpha\cos\omega - \beta\sin\omega)^2 - (a^2\cos^2\omega + b^2\sin^2\omega)\ldots(21).$$

Now (21), it is easy to see, expresses that the two tangents of F intersect on the conic

$$F + \frac{\theta}{k^2}J = 0 \ldots\ldots\ldots\ldots\ldots\ldots(22),$$

where $J = p^2 - k^2$. But Dr. Casey has shown that the quartic can be generated in the manner described in four ways, the four conics such as F being confocal. He has further shown that, if λ, μ, ν be the perpendiculars from three centres of inversion on a line, then F and J can be written tangentially in the form

$$F = (a_1^2 - a_4^2)\frac{\lambda^2}{k_1^2} + (a_2^2 - a_4^2)\frac{\mu^2}{k_2^2} + (a_3^2 - a_4^2)\frac{\nu^2}{k_3^2}\ldots(23),$$

$$J = -k_4^2\left\{\frac{\lambda^2}{k_1^2} + \frac{\mu^2}{k_2^2} + \frac{\nu^2}{k_3^2}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(24),$$

where a_1, a_2, a_3, a_4 are the halved transverse axes of the four focal conics F, &c., and k_1, k_2, k_3, k_4 are the radii of the four circles such as J. The conic (22) thus becomes

$$\Sigma = (a_1^2 - \phi)\frac{\lambda^2}{k_1^2} + (a_2^2 - \phi)\frac{\mu^2}{k_2^2} + (a_3^2 - \phi)\frac{\nu^2}{k_3^2}\ldots.(25),$$

where $\phi = a_4^2 + \theta$. Now to a pair of points, inverse with regard to J, corresponds a point on the conic F, as we see by the mode of generation, so that we have proved that if the line joining two points P, Q on the quartic touches the inscribed conic (4), then the corresponding points on the conic F are such that the tangents at them intersect on the conic (25). Conversely, however, if the tangents of F intersect on (25),

the points on the quartic must be properly selected, that is, the line joining any pair of corresponding points will not touch an inscribed conic. Now for du we have from (3)

$$du = \frac{\rho d\omega}{\frac{dV}{d\rho}} = \frac{d\omega}{p\sqrt{(p^2 - k^2)}} \quad \ldots\ldots\ldots\ldots (26).$$

But if ψ is the eccentric angle of the point on F, we have

$$d\omega = \frac{ab d\psi}{a^2 \sin^2\psi + b^2 \cos^2\psi},$$

and

$$p = \frac{b\alpha \cos\psi + a\beta \sin\psi - ab}{\sqrt{(a^2 \sin^2\psi + b^2 \cos^2\psi)}} \quad \ldots\ldots\ldots\ldots (27),$$

so that we have

$$du = \frac{d\psi}{\sqrt{\{(b\alpha \cos\psi + a\beta \sin\psi - ab)^2 - k^2 (a^2 \sin^2\psi + b^2 \cos^2\psi)\}}} \quad \ldots\ldots\ldots (28),$$

and if $du_1 = du_2 = 0$, then the tangents to F at u_1, u_2 intersect on $F + \frac{\theta}{k^2} J = 0$, but the line joining u_1, u_2 on the quartic touches an inscribed conic only if $d(u_1 + u_2) = 0$.

Now if we have the relation $u_1 + u_2 + u_3 + u_4 = \omega$, taken by itself, connecting four points on the quartic, these points are concyclic. Hence the points of contact u_1, u_2 of a circle having double contact with the quartic are such that

$$2(u_1 + u_2) = \omega \quad \ldots\ldots\ldots\ldots (29),$$

which by the theory of elliptic integrals gives the four modes of generation. Now for a quadrliateral we have from (15) $4\sigma = 2\omega$, which gives sixteen values of σ. But four of these correspond to $2\sigma = \omega$, which from (29) corresponds to the points of contact of a circle having double contact with the curve, viz. a pair of points collinear with a centre of inversion. Again σ and $\omega - \sigma$ correspond to the same conic of the system (4), so that the equation (16) gives six solutions, as we can also prove as follows. If the conics F and Σ are such that an infinite number of quadrilaterals can be circumscribed about F so as to be inscribed in Σ, then we have

$$\frac{a_1^2 - a_4^2}{a_1^2 - \phi} + \frac{(a_2^2 - a_4^2)}{a_2^2 - \phi} - \frac{(a_3^2 - a_4^2)}{a_3^2 - \phi} = 0 \quad \ldots\ldots (30),$$

and two other similar relations; for in the case of two conics expressed tangentially in the forms

$$F=\kappa_1\lambda^2+\kappa_2\mu^2+\kappa_3\nu^2,\quad \Sigma=\lambda^2+\mu^2+\nu^2 \quad \ldots\ldots(31),$$

the invariant condition is

$$(\kappa_2+\kappa_3-\kappa_1)\,(\kappa_3+\kappa_1-\kappa_2)\,(\kappa_1+\kappa_2-\kappa_3)=0.$$

Now (30) can be written

$$\frac{1}{\phi-a_1^2}+\frac{1}{\phi-a_2^2}-\frac{1}{\phi-a_3^2}-\frac{1}{\phi-a_4^2}=0 \quad \ldots.(32),$$

and the values of ϕ given by this relation are a pair of conjugate roots of the sextic covariant of

$$\Phi=(\phi-a_1^2)\,(\phi-a_2^2)\,(\phi-a_3^2)\,(\phi-a_4^2) \quad \ldots.(33).$$

We see thus that the six conics corresponding to inscribed quadrilaterals are determined by the sextic covariant of the biquadratic (33). Again, if each side of the quadrilateral touches a different conic of the system (4), these four conics must be such that

$$\Sigma_4'\frac{\sqrt{\phi\,(a_i^2)}}{f'(a_i^2)}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(34),$$

where $\phi\,(a_1^2)=(a_1^2-\phi_1)\,(a_1^2-\phi_2)\,(a_1^2-\phi_3)\,(a_1^2-\phi_4)$,

$$f(a^2)=(a^2-a_1^2)\,(a^2-a_2^2)\,(a^2-a_3^2)\,(a^2-a_4^2).$$

If two sides touch one conic and two sides another conic, it is easily seen hence that these two conics must be such that the values of ϕ_1, ϕ_2 in (25) are harmonically connected with a pair of conjugate roots of the sextic covariant. But it may be observed that we can have an infinite number of quadrilaterals without any relation connecting the conics; for there can be a singly infinite number of values of u_1, u_2, u_3, u_4 subject to the relations

$$u_1+u_2=\sigma_1,\ u_2+u_3=\sigma_2,\ u_3+u_4=\omega-\sigma_1,\ u_4+u_1=\omega-\sigma_2 \quad \ldots\ldots\ldots(35).$$

In this case the vertices of the quadrilateral lie on a circle, as we have seen, or as can easily be proved from (3); for, equating (4) to R^2-PQ and introducing the equation of the quartic, we see that PQ, namely, a pair of tangents of the conic, meet the quartic on two circles.

Proceeding now to the case of the hexagon, we have seen

that in this case $6\sigma = 3\omega$. This includes the four irrelevant values derived from $2\sigma = \omega$. We find thus 16 conics of the system (4) satisfying the condition. Since in this case

$$u_1 + u_4 = 3\sigma - \omega = \tfrac{1}{2}\omega,$$

it follows that u_1, u_4 are inverse points with regard to one of the four centres of inversion, and the same is the case with regard to the pairs of points u_2, u_5; u_3, u_6. Hence, corresponding to the hexagon inscribed in the quartic there is a triangle circumscribed about F and inscribed in Σ. Now the invariant relation connecting the conics (23) and (24) in this case is

$$\sqrt{\kappa_1} + \sqrt{\kappa_2} + \sqrt{\kappa_3} = 0.$$

Hence we have

$$\sqrt{\left(\frac{a_1^2 - a_4^2}{\phi - a_1^2}\right)} + \sqrt{\left(\frac{a_2^2 - a_4^2}{\phi - a_2^2}\right)} + \sqrt{\left(\frac{a_3^2 - a_4^2}{\phi - a_3^2}\right)} = 0 \ldots (36),$$

which gives four values of ϕ.

We have also the three other similar equations corresponding to the three other centres of inversion, viz.

$$\left.\begin{aligned}
&\sqrt{\left(\frac{a_1^2 - a_3^2}{\phi - a_1^2}\right)} + \sqrt{\left(\frac{a_2^2 - a_3^2}{\phi - a_2^2}\right)} + \sqrt{\left(\frac{a_4^2 - a_3^2}{\phi - a_4^2}\right)} = 0 \\
&\sqrt{\left(\frac{a_1^2 - a_2^2}{\phi - a_1^2}\right)} + \sqrt{\left(\frac{a_3^2 - a_2^2}{\phi - a_3^2}\right)} + \sqrt{\left(\frac{a_4^2 - a_2^2}{\phi - a_4^2}\right)} = 0 \\
&\sqrt{\left(\frac{a_2^2 - a_1^2}{\phi - a_2^2}\right)} + \sqrt{\left(\frac{a_3^2 - a_1^2}{\phi - a_3^2}\right)} + \sqrt{\left(\frac{a_4^2 - a_1^2}{\phi - a_4^2}\right)} = 0
\end{aligned}\right\} \ldots (37).$$

If the sides of the hexagon touch in pairs three different conics of the system, it can be shown that these three conics must be such that

$$(a_2^2 - a_3^2)\sqrt{\left\{\frac{\phi(a_1^2)}{a_1^2 - a_4^2}\right\}} + (a_3^2 - a_1^2)\sqrt{\left\{\frac{\phi(a_2^2)}{a_2^2 - a_4^2}\right\}} + (a_1^2 - a_2^2)\sqrt{\left\{\frac{\phi(a_3^2)}{a_3^2 - a_4^2}\right\}} = 0 \ldots\ldots (38),$$

where $$\phi(a^2) = (a^2 - \phi_1)(a^2 - \phi_2)(a_2 - \phi_3),$$

or one of three other similar conditions obtained by interchanging a_1, a_2, a_3, a_4.

I now proceed to consider the inscribed octagon. For

such a figure it can be shown that the conics F and Σ (23) must be such that

$$\frac{1}{(\kappa_2+\kappa_3-\kappa_1)^2}+\frac{1}{(\kappa_3+\kappa_1-\kappa_2)^2}=\frac{1}{(\kappa_1+\kappa_2-\kappa_3)^2} \quad \ldots(39),$$

or one of the two other similar conditions obtained by interchanging κ_1, κ_2, κ_3. Hence, if J_1, J_2, J_3 are the quadratic factors of the sextic covariant of

$$\Phi=(\phi-a_1^2)(\phi-a_2^2)(\phi-a_3^2)(\phi-a_4^2),$$

we must have

$$\left.\begin{aligned} &\frac{1}{J_1^2}+\frac{1}{J_2^2}-\frac{1}{J_3^2}=0 \\ \text{or}\quad &\frac{1}{J_1^2}-\frac{1}{J_2^2}+\frac{1}{J_3^2}=0 \\ \text{or}\quad &\frac{1}{J_1^2}-\frac{1}{J_2^2}-\frac{1}{J_3^2}=0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(40),$$

where, it may be observed that, identically,

$$(a_1^2-a_2^2)(a_3^2-a_4^2)J_1^2+(a_3^2-a_1^2)(a_2^2-a_4^2)J_2^2 + (a_2^2-a_3^2)(a_1^2-a_4^2)J_3^2=0 \quad \ldots\ldots(41).$$

We thus obtain 24 conics altogether in this case.

It may be observed that, if e_1, e_2, e_3 are the roots of $4c^3-Sc+T=0$, where S, T are the invariants of Φ, we have $c_1\Phi-H=J_1^2$, &c., so that (40) gives

$$\frac{1}{z-c_1}+\frac{1}{z-c_2}=\frac{1}{z-c_3},$$

whence

$$z^2-2c_3z+\tfrac{1}{4}S=2c_3^2 \quad \ldots\ldots\ldots\ldots\ldots(42),$$

where H is the Hessian of Φ and $z=\dfrac{H}{\Phi}$.

We thus obtain the equation of the 24th degree in the form of a covariant of Φ.

It can be shown that, if the sides of an inscribed octagon touch in pairs four different conics of the system, then the relation connecting ϕ_1, ϕ_2, ϕ_3, ϕ_4 must be found by eliminating p, q, r, s from the terms of the identity

$$(x-\phi_1)(x-\phi_2)(x-\phi_3)(x-\phi_4)=(p+qx)^2(x-a_1^2)(x-a_2^2) + (rx+s)^2(x-a_3^2)(x-a_4^2) \quad \ldots\ldots\ldots(43),$$

or one of the two similar equations obtained by interchanging a_1, a_2, a_3, a_4.

In the case of the decagon it is readily seen that the ten vertices form five pairs of points inverse with regard to one of the circles of inversion. Thus the corresponding figure on one of the conics F is a pentagon. Hence we have to express the condition that pentagons circumscribed about F should be inscribed in Σ. We then take the three other conditions corresponding to the three other centres of inversion.

There are 48 conics in this case, as we have $10\sigma = 5\omega$ which includes $2\sigma = \omega$, the four values corresponding to which must be neglected; that is, $\frac{1}{2}(100-4) = 48$. The actual values of ϕ will be found by eliminating $\lambda, \mu, \nu, \rho, \sigma$ from the terms of the identity

$$(x-\phi)^6 = (\lambda+\mu x+\nu x^2)^2(x-a_1^2) + (\rho x+\sigma)^2(x-a_2^2)(x-a_3^2)(x-a_4^2). \quad \ldots.(44),$$

or one of three similar equations obtained by interchanging a_1, a_2, a_3, a_4. Proceeding to the case of the dodecagon we have $12\sigma = 6\omega$.

The solutions of this corresponding to $6\sigma = 3\omega$ and $4\sigma = 2\omega$ must be neglected, the four solutions common to these equations being counted once. We thus see that there are 48 conics.

The corresponding values of ϕ will be found, by a simple application of Abel's theorem, from the terms of the identity

$$(x-\phi)^4 = (\lambda+\mu x+\nu x^2)^2(x-a_1^2)(x-a_2^2) + (\lambda'+\mu' x+\nu' x^2)^2(x-a_3^2)(x-a_4^2) \ldots..(45)$$

by the elimination of $\lambda, \mu, \nu, \lambda', \mu', \nu'$, or from one of the two similar equations obtained by interchanging a_1, a_2, a_3, a_4.

With the usual notation of elliptic functions one of these relations is $k' \operatorname{sn} u = (1 - \operatorname{sn} u)\operatorname{dn} u$. From this we readily find

$$(J_1^2J_2^2 + J_2^2J_3^2 + J_3^2J_1^2)^2 - 4J_1^2J_2^2J_3^2(J_1^4 + J_2^4 + J_3^4) \ldots(46),$$

observing that $J_1 : J_2 : J_3 = \operatorname{cn} u : \operatorname{dn} u : 1$.

Putting $H = \Phi z$, we get from this

$$(3z^2 - \tfrac{1}{4}S)^2 - 4(z-c_2)(z-c_3)(2z^2 - c_1 z + c_1^2 + c_2c_3) = 0 \ldots(47),$$

which gives four values of z, and therefore sixteen values of ϕ.

The other 32 values of ϕ are evidently given by

$$\left.\begin{aligned}(3z^2 - \tfrac{1}{4}S)^2 - 4(z-c_3)(z-c_1)(2z^2 - c_2 z + c_2^2 + c_1c_3) = 0\\ (3z^2 - \tfrac{1}{4}S)^2 - 4(z-c_1)(z-c_2)(2z^2 - c_3 z + c_3^2 + c_1c_2) = 0\end{aligned}\right\} \ldots(48).$$

In the preceding cases it may be observed that the alternate diagonals, namely the lines joining the points u_1, u_4; u_1, u_6, &c., touch inscribed conics of the system, except in the cases in which they pass through a centre of inversion. For the other diagonals, such as the line joining u_1, u_2, there is $u_1 - u_2 = a$ constant, and the envelope is a curve of the eighth class, except in the case in which the constant is half a complete period of the elliptic function, when the envelope reduces to a curve of the fourth class.

If the bicircular quartic has a finite node, two of the focal conics H coincide and the biquadric (33) has a square factor. In this case there is a considerable reduction in the number of the solutions. Thus, if (33) be written

$$\Phi = (\phi - a^2)^2 (\phi - a_1^2)(\phi - a_2^2),$$

there is but one conic in the case of the quadrilateral, given by

$$\phi (a_1^2 + a_2^2 - 2a^2) = 2a_1^2 a_2^2 - a^2 (a_1^2 + a_2^2) \ldots\ldots (49),$$

and for the hexagon there are only two conics given by

$$\left.\begin{aligned} \phi (a_1^2 + 3a_2^2 - 4a^2) &= 4a_1^2 a_2^2 - 3a^2 a_1^2 - a^2 a_2^2 \\ \phi (a_2^2 + 3a_1^2 - 4a^2) &= 4a_1^2 a_2^2 - 3a^2 a_2^2 - a_2 a_1^2 \end{aligned}\right\} \ldots (50).$$

In the case of the Cartesian oval the generating conic F is a circle and the circular points are cusps. The conic (4) then becomes

$$\theta^2 + \theta (x^2 + y^2 - 2\alpha x - 2\beta y + k^2) + a^2 (x^2 + y^2) = 0 \ldots (51),$$

that is, a circle having double contact with the curve at points symmetrically placed with regard to the axis. The discriminant Φ of this circle is

$$\phi (\phi - a_1^2)(\phi - a_2^2)(\phi - a_3^2) \quad \ldots\ldots\ldots (52),$$

where $\theta + a^2 = \phi$, and a_1, a_2, a_3 are the radii of three circles concentric with the triple focus and passing through the points of contact with the curve of the tangents drawn from the three collinear foci, respectively. Since this expression merely differs from (33) by making $a_4 = 0$, which does not affect the degree of any covariant of Φ, we have precisely the same number of circles, touching the sides of inscribed polygons, as there are inscribed conics in the general case of the bicircular quartic.

If the two nodes of the quartic coincide and form a tacnode cusp, there is a similar system of inscribed conics.

Taking the tacnode at infinity on $x=0$ and the line at infinity as the tangent thereat, the curve may be written

$$y^2=(x-a)(x-b)(x-c)(x-d) \quad \ldots\ldots(53),$$

and if, further, the origin be taken at the centroid of the four points in which the curve meets the axis of symmetry $y=0$, we may write the curve

$$V=y^2-x^4+ax^2+2gx+c=0 \quad \ldots\ldots(54),$$

which is the envelope of

$$U=\theta^2+2\theta x^2+y^2+ax^2+2gx+c=0 \ldots\ldots(55).$$

The curve then being capable of being written in the form $U=(\theta+x^2)^2$, it is easily shown that a tangent of U meets the quartic V in a pair of points such that $u+u_2=$ a constant, where

$$u=\int\frac{dx}{\sqrt{(x^4-ax^2-2gx-c)}} \quad \ldots\ldots(56).$$

In this case the inscribed conics are found from the covariants of the discriminant of U instead of Φ the discriminant of (4). This discriminant is

$$2\theta^3+a\theta^2+2c\theta+ac-g^2 \quad \ldots\ldots(57),$$

and differs from Φ (33) in having one of its roots infinite.

This does not affect the number of the roots of the covariants, as we have still to consider (57) as a biquadratic expression of which one factor corresponds to infinity, so that we have precisely the same number of inscribed conics for polygons of a given even number of sides, as we have in the general case of the binodal quartic.

ON THE PLANE QUARTIC CURVE WITH A CENTRE AND THE CORRESPONDING CONE.

By *R. A. Roberts.*

By a centre in the general case I mean a point which becomes a centre when the curve is projected in a certain way, a centre then being such that the curve after having been turned round it through an angle of 180° occupies the same position. A centre implies also the existence of an axis on which lies one of the double points of an involution determined by the four points in which a line passing

through the centre meets the curve. Taking the axis as the line z and the centre at the point xy, we may write the quartic

$$z^4 + 2z^2u_2 + u_4 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where

$$u_2 = (a, b, c)(x, y)^2, \quad u_4 = (a', b', c', d', e')(x, y)^4.$$

We see thus that the curve does not involve more than 12 constants, so that it differs from a general quartic, which involves 14 constants, by two conditions.

I proceed to investigate some properties of this curve, especially in relation to certain conics connected with it. The equation (1) may be written

$$(z^2 + u_2)^2 = u_2^2 - u_4 \quad \ldots\ldots\ldots\ldots\ldots (2),$$

so that if we choose x, y as a pair of conjugate factors of the sextic covariant of $u_2^2 - u_4$, we have

$$U = (z^2 + ax^2 + by^2 + 2cxy)^2 - (x^4 + y^4 + 6mx^2y^2) = 0 \ldots (3);$$

and there are three such reductions corresponding to the three canonical forms of $u_2^2 - u_4$. If, however, $u_2^2 - u_4$ has a square factor, then in a single way it can be reduced to the form $x^2(x^2 + y^2)$, and the curve is

$$U = (z^2 + ax^2 + by^2 + 2cxy)^2 - x^2(x^2 + y^2) = 0 \quad \ldots (4).$$

The latter curve is a binodal quartic with a centre; it contains eleven constants, thus having one constant less than the general binodal quartic.

Now for convenience we may without loss of generality take $c = -1$ in (3). Putting then $x^2 = X$, $y^2 = Y$, $z^2 - 2xy = Z$, we get (3) in the form

$$(Z + aX + bY)^2 - (X^2 + Y^2 + 6mXY) = 0 \quad \ldots (5),$$

which represents a general conic in X, Y, Z.

Considering then the conic

$$z^2 - 4xy = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and writing a tangent in the form

$$\mu^2 x - \mu z + y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

we have, for the intersection of two tangents μ_1, μ_2,

$$x : y : z = 1 : \mu_1\mu_2 : (\mu_1 + \mu_2) \quad \ldots\ldots\ldots\ldots\ldots (8).$$

Hence considering the transformation according to which

we substitute for μ_1, μ_2 their squares, we have for the corresponding point

$$X:Y:Z=1:\mu_1^2\mu_2^2:(\mu_1^2+\mu_2^2)=x^2:y^2:(z^2-2xy)..(9);$$

that is, the substitution is the same as that by which the conic (5) is transformed into the quartic (3). Hence the conic (6) bears a permanent relation to the quartic, and since there are three reductions to the form (3), there are three such conics. For the binodal quartic (4) there is one such conic. And it is be observed that in a certain case there are six such conics. Suppose $c=0$ in (3), then the curve can be written in the unique canonical form

$$(a,\ b,\ c,\ f,\ g,\ h)(x^2,\ y^2,\ z^2)^2=0 \ldots\ldots\ldots(10),$$

and has three centres. There are then two related conics corresponding to each centre, so that there are six related conics altogether, as just stated.

By the substitution (10) a tangent (7) to the conic (6) is transformed into a pair of tangents of the same conic intersecting on the axis z, and corresponding to a point $\mu_1=a$, $\mu_2=b$, there are four points $\mu_1=\pm\sqrt{a}$, $\mu_2=\pm\sqrt{b}$. These latter points evidently form a quadrilateral circumscribed about the conic (6) and having two of its intersections of sides on the axis.

Hence there are an infinite number of quadrilaterals inscribed in the quartic (3) and circumscribed about one of the three conics such as (6).

I now investigate the double tangents of the curve. From (3) it is evident that $u_2^2-u_4=0$ gives four double tangents of the curve passing through the centre. These four lines correspond to the two tangents drawn from the centre to the conic (5).

Again it is evident that four pairs of double tangents correspond to the four common tangents of the conics (5) and (6). Hence from the three conics such as (6) and the three corresponding conics such as (5) we get twelve pairs of double tangents. These twelve pairs together with the four double tangents passing through the centre comprise the entire 28 double tangents of the curve. With regard to the inflexional tangents of the curve there is no such simplicity, but we can investigate algebraically relations that must be satisfied by the coordinates l, m, n of these lines. Suppose the line

$$lx+my+nz=0 \text{ or } l+m\mu_1\mu_2+n(\mu_1+\mu_2)=0\ldots(11)$$

to be an inflexional tangent of the quartic

$$(a,\ b,\ c,\ f,\ g,\ h)(1,\ \mu_1^2\mu_2^2,\ \mu_1^2+\mu_2^2)^2=0 \ \ldots(12).$$

Now (11) can be converted into

$$\{l^2+m^2\mu_1^2\mu_2^2-n^2(\mu_1^2+\mu_2^2)\}^2-4(n^2-lm)^2\mu_1^2\mu_2^2=0.$$

Hence by the substitution of μ_1, μ_2 for μ_1^2, μ_2^2 respectively, the conics

$$\{l^2+m^2\mu_1\mu_2-n^2(\mu_1+\mu_2)\}-4(n^2-lm)^2\mu_1\mu_2=0\ldots(13),$$

$$(a,\ b,\ c,\ f,\ g,\ h)\,(1,\ \mu_1\mu_2,\ \mu_1+\mu_2)^2=0\ \ldots\ldots(14)$$

must osculate. Expressing these conics in x, y, z coordinates and then taking their tangential equations we get

$$\Sigma=(2lm-n^2)\nu^2+l^2\mu\nu+m^2\nu\lambda+n^2\lambda\mu=0\ \ldots(15),$$

$$\Sigma'=(A,\ B,\ C,\ F,\ G,\ H)\,(\lambda,\ \mu,\ \nu)^2=0\ \ldots(16),$$

and the conditions that these conics should osculate give the relations that must be satisfied by the coordinates l, m, n of the inflexional tangents. Writing the discriminant of $k\Sigma'+\Sigma$ in the form $\Delta k^3+\Theta k^2+\Theta' k+\Delta'=0$, we have for osculation

$$\Theta^2-3\Delta\Theta'=0,\ \Theta\Theta'-9\Delta\Delta'=0,\ \Theta'^2-3\Delta'\Theta=0\ldots(17).$$

The first two of these equations give relations of the fourth and sixth degrees respectively in l, m, n, and these determine the 24 inflexional tangents.

It may be worth while noticing that the investigation of the 24 inflexional tangents of the tricentral quartic can be made to depend upon the same equations as determine the six inflexional tangents of a trinodal quartic. Let

$$lx+my+nz=0$$

be an inflexional tangent of the quartic

$$(a,\ b,\ c,\ f,\ g,\ h)\,(x^2,\ y^2,\ z^2)=0\ \ldots\ldots\ldots(18),$$

then x, y, z being substituted for x^2, y^2, z^2 respectively it is evident that the conics

$$\left.\begin{aligned} &l\sqrt{x}+m\sqrt{y}+n\sqrt{z}=0\\ &(a,\ b,\ c,\ f,\ g,\ h)\,(x,\ y,\ z)^2=0\end{aligned}\right\}\ \ldots\ldots\ldots(19)$$

must osculate. Hence, taking the tangential equations,

$$\left.\begin{aligned} &l^2\mu\nu+m^2\nu\lambda+n^2\lambda\mu=0\\ &(A,\ B,\ C,\ F,\ G,\ H)\,(\lambda,\ \mu,\ \nu)^2=0\end{aligned}\right\}\ \ldots\ldots(20)$$

must osculate. Now let us put $\lambda:\mu:\nu = yz:zx:xy$; then it is evident that the line $l^2x + m^2y + n^2z = 0$ must be an inflexional tangent of the trinodal quartic

$$Ay^2z^2 + Bz^2x^2 + Cx^2y^2 + 2xyz(Fx + Gy + Hz) = 0 \text{ ..(21)};$$

that is, the problem of finding the 24 inflexional tangents of the quartic (18) is made to depend algebraically upon the problem of finding the six inflexional tangents of trinodal quartics (21), four tangents of (18) corresponding to each one of (21).

It may be observed that this process is a particular case of a more general result, viz. that if a quartic curve be generated as the locus of a point whose polar line with regard to a cubic U touches a conic V, then the inflexional tangents of the quartic can be found by expressing that the polar conic of a *line* with regard to U osculates the conic V. The case under consideration corresponds to the special cubic which can be expressed as the sum of three cubes. That is, if

$$U = x^3 + y^3 + z^3 + 6mxyz = 0,$$

$$V = (a, b, c, f, g, h)(x, y, z)^2 = 0,$$

the quartic is

$$(A, B, C, F, G, H)(x^2 + 2myz,\ y^2 + 2mzx,\ z^2 + 2mxy)^2 = 0 \quad \text{.....(22)}$$

and if $\alpha x + \beta y + \gamma z = 0$ is an inflexional tangent, the conic

$$\alpha^2(yz - m^2x^2) + \beta^2(zx - m^2y^2) + \gamma^2(xy - m^2z^2)$$
$$+ 2\beta\gamma(m^2yz - m^2x^2) + 2\gamma\alpha(m^2zx - my^2) + 2\alpha\beta(m^2xy - mz^2) = 0 \quad \text{.....(23)},$$

must osculate the conic V, the conditions being those at (17). The quartic (18) corresponds to the case of $m = 0$.

I now proceed to show that if the quartic satisfies a certain invariant condition, it is connected with one of the conics (6) by a certain geometrical relation. We know that if a certain invariant relation connects two conics, then an infinite number of polygons of n sides can be circumscribed about one conic and inscribed in the other. Let these two conics be (6) and (5), then transforming by the substitution (9) we see that if an invariant condition is satisfied, there will be an infinite number of polygons of n sides circumscribed about the conic

CONTENTS.

PAGE

On polygons inscribed in a binodal quartic and circumscribed about a conic. By R. A. ROBERTS - - - - - - - - - - - - 161

On the plane quartic curve with a centre and the corresponding cone. By R. A. ROBERTS - - - - - - - - - - - - - 171

Papers for insertion should be sent to Dr. Glaisher, Trinity College, Cambridge, or to Messrs. Metcalfe and Co. Limited, Printing Office, Trinity Street, Cambridge.

An index to vols. i—xxv has been published. It is issued gratis to subscribers, but may be obtained separately, price Two Shillings.

No. CCCCVIII.] NEW SERIES. [April, 1905.

Stanford University Library JUN 6 1905

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XXXIV.—NO. 12.

MACMILLAN AND CO.
London and Cambridge.
1905.

Price—One Shilling.

METCALFE AND CO. LIMITED, CAMBRIDGE.

Now from (30) we see that the four tangents corresponding to $\phi(\mu) - k = 0$ intersect the four tangents corresponding to $1 + k\phi(\mu) = 0$ on the quartic, as is also evident from the fact that (30) can be written

$$(1 + k\phi_1)(1 + k\phi_2) + (\phi_1 - k)(\phi_2 - k) = 0 \quad \ldots.(32),$$

where k is arbitrary. These sixteen points can be arranged so as to form quadrilaterals, hexagons, and octagons, so that the writing of the quartic in the form (29) gives a solution corresponding to those figures. From (32) we see that the intersections of alternate sides of the polygons satisfy

$$\frac{\phi_1 - k}{1 + k\phi_1} - \frac{\phi_3 - k}{1 + k\phi_3} = 0, \text{ or } \phi_1 - \phi_3 = 0 \quad \ldots\ldots(33),$$

which is a given locus consisting of the axis z and a cubic with a point of inflexion at xy and the axis z as the corresponding harmonic polar.

Again suppose we have,

$$(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)(z - \alpha_4)(z - \alpha_5) + (z + \alpha_1)(z + \alpha_2)(z + \alpha_3)(z + \alpha_4)(z + \alpha_5) = 0 \ldots(34).$$

This is divisible by z and and the remainder being of the form

$$z^4 + u_2 z^2 + u_4 = 0$$

represents a quartic with a centre. Now, as at (30), (34) becomes of the form

$$\phi(\mu_1)\,\phi(\mu_2) - 1 = 0 \quad \ldots\ldots\ldots\ldots(35),$$

where

$$\phi(\mu) = \frac{(\mu - a_1) \ldots (\mu - a_5)}{(\mu + a_1) \ldots (\mu + a_5)} \quad \ldots\ldots\ldots\ldots(36).$$

Hence similarly we have an infinite system of 25 points lying on the axis and the curve, and since 5 of the points lie on the axis, 20 points are on the curve. These 20 points can be arranged so as to form figures of 4, 6, 8, and 10 sides, and the reduction of the quartic to the form (35) gives a solution of the problem of the polygons corresponding to these figures. Also, for the intersection of the alternate sides of these figures, we have

$$\phi(\mu_1) - \phi(\mu_3) = 0 \quad \ldots\ldots\ldots\ldots(37),$$

which represents a quartic with a centre. This equation (37) can by resolution into partial fractions evidently be written

CONTENTS.

PAGE

On polygons inscribed in a binodal quartic and circumscribed about a conic. By R. A. ROBERTS - - - - - - - - - - - - - 161

On the plane quartic curve with a centre and the corresponding cone. By R. A. ROBERTS - - - - - - - - - - - - - 171

Papers for insertion should be sent to Dr. Glaisher, Trinity College, Cambridge, or to Messrs. Metcalfe and Co. Limited, Printing Office, Trinity Street, Cambridge.

An index to vols. i—xxv has been published. It is issued gratis to subscribers, but may be obtained separately, price Two Shillings.

No. CCCCVIII.] NEW SERIES. [April, 1905.

Stanford University Libraries JUN 6 1905

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XXXIV.—NO. 12.

MACMILLAN AND CO.
London and Cambridge.
1905.

Price—One Shilling.

MACMILLAN AND BOWES, CAMBRIDGE.

in the form

$$\Sigma'^{-5}\frac{A_r}{z-\alpha_r}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (38),$$

or the equivalent form

$$\Sigma'^{-5}\frac{B_r}{z+\alpha_r}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (39).$$

These equations represent a locus passing through the 10 points of intersection of five tangents to the conic. It is known otherwise that when a quartic is written in the form (38), then it is capable of being written in this form in a singly infinite number of ways, as it is then the covariant of a special quartic, viz. a quartic which is capable of being expressed as the sum of five squares (see Salmon's *Higher Plane Curves*). The 10 points referred to can evidently be arranged so as to form triangles, quadrilaterals, and pentagons. Thus the reduction of the central quartic to the form (38) gives a solution of the problem of the polygons in the three cases just mentioned.

I now proceed to consider the corresponding cones. The special case most worthy of attention is when the conic (6) coincides with the cone standing on the imaginary circle at infinity. Taking rectangular coordinates the conic (6) will coincide with

$$x^2+y^2+z^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (40)$$

if for x, y, z we substitute

$$(x+iy), -(x-iy), 2z,$$

so that the transformation (9) becomes

$$X:Y:Z=i(x^2-y^2):2ixy:(x^2+y^2+2z^2) \quad \ldots (41).$$

The quartic then being a function of the second degree in X, Y, Z is a function of the second degree in x^2+z^2, y^2+z^2, $2xy$, so that it can be generated as the envelope of the cones

$$U=\alpha(x^2+z^2)+\beta(y^2+z^2)+2\gamma xy=0 \quad \ldots\ldots (42),$$

where

$$(a, b, c, f, g, h)(\alpha, \beta, \gamma)^2=0 \quad \ldots\ldots\ldots (43).$$

Now when

$$\alpha\beta-\gamma^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (44),$$

the cone (42) breaks up into two planes touching the imaginary circle at infinity and intersecting in a line on the plane x. Thus (43) and (44), taken together, determine four bitangent focal lines of the quartic lying in the plane z. Again, from (43), we may take α, β, γ proportional to quadratic expressions in a parameter θ. The equation (42) then becomes

$$V=\theta^2u_1+\theta u_2+u_3=0 \quad \ldots\ldots\ldots\ldots\ldots (45).$$

It is easy to see, then, that if we take three such cones the quartic $u_2^2 - u_1 u_3 = 0$ can be written

$$l\sqrt{V_1} + m\sqrt{V_2} + n\sqrt{V_3} = 0 \quad \ldots\ldots\ldots\ldots(46),$$

where l, m, n are constants. Hence, considering the four bifocal lines, and observing that V then is proportional to $\sin^2\rho$, where ρ is the angle a variable edge of the quartic cone makes with a bifocal line, we must have

$$l\sin\rho_1 + m\sin\rho_2 + n\sin\rho_3 = 0 \quad \ldots\ldots\ldots(47),$$

$$l'\sin\rho_1 + m'\sin\rho_2 + n'\sin\rho_4 = 0 \quad \ldots\ldots\ldots(48).$$

Now let us consider two cones (42) V_1, V_2, then we have

$$\frac{dV_1}{dx}\frac{dV_2}{dx} + \frac{dV_1}{dy}\frac{dV_2}{dy} + \frac{dV_1}{dz}\frac{dV_2}{dz} = 2\{(\alpha_2+\beta_2)V_1 + (\alpha_1+\beta_1)V_2 - (\alpha_1\beta_2 + \beta_1\alpha_2 - 2\gamma_1\gamma_2)(x^2+y^2)\},$$

so that V_1, V_2 cut orthogonally if

$$\alpha_1\beta_2 + \beta_1\alpha_2 - 2\gamma_1\gamma_2 = 0 \quad \ldots\ldots\ldots\ldots(49),$$

in which case, it may be observed, the pairs of cyclic planes intersecting in xy, viz.

$$\alpha_1 y^2 + \beta_1 x^2 - 2\gamma_1 xy = 0, \quad \alpha_2 y^2 + \beta_2 x^2 - 2\gamma_2 xy = 0,$$

form a harmonic pencil. Now let us form the condition that

$$lV_1 + mV_2 + nV_3 = 0 \text{ and } l'V_1 + m'V_2 + n'V_3 = 0$$

should cut orthogonally. Taking

$$\alpha_1\beta_1 - \gamma_1^2 = \alpha_2\beta_2 - \gamma_2^2 = \alpha_3\beta_3 - \gamma_3^2$$

for convenience, we find

$$ll' + mm' + nn' = 0 \ldots\ldots\ldots\ldots\ldots(50).$$

Again, let us form the condition that

$$\lambda\sin^2\rho_1 + \mu\sin^2\rho_2 + \nu\sin^2\rho_3 = 0, \quad \lambda'\sin^2\rho_1 + \mu'\sin^2\rho_2 + \nu'\sin^2\rho_3 = 0 \quad \ldots\ldots(51)$$

should cut orthogonally, where $\sin^2\rho = (x\cos\theta + y\sin\theta)^2 + z^2$. We find

$$(\mu\nu' + \mu'\nu)\sin^2 a + (\nu\lambda' + \nu'\lambda)\sin^2 b + (\lambda\mu' + \lambda'\mu)\sin^2 c = 0 \quad \ldots\ldots(52),$$

where a, b, c are the angles between the pairs of lines θ_1, θ_2, θ_3. Writing now r, s, t for $\sin\rho_1$, $\sin\rho_2$, $\sin\rho_3$ respectively, the quartic (47) becomes

$$lr + ms + nt = 0 \ldots\ldots\ldots\ldots\ldots(53),$$

and a cone of the second order touching it at $r!\,s!\,t!$ is evidently

$$\frac{lr^2}{r'}+\frac{ms^2}{s'}+\frac{nt^2}{t'}=0 \ldots\ldots\ldots\ldots\ldots\ldots(54).$$

Hence, if two touching cones for two quartics such as (53) cut orthogonally, we have from (52)

$$\sin^2 a\,(mn'+m'n)\,r'+\sin^2 b\,(nl'+n'l)\,s' + \sin^2 c\,(lm'+l'm)\,t'=0 \ldots(54^*);$$

but from (53)

$$r' : s' : t' = mn'-m'n : nl'-n'l : lm'-l'm,$$

so that we obtain

$$\sin^2 a\,(m^2n'^2-m'^2n^2)+\sin^2 b\,(n^2l'^2-n'^2l^2)+\sin^2 c\,(l^2m'^2-l'^2m^2)=0 \ldots\ldots(55).$$

Now this is the condition that the two quartics should have their fourth focal line in common, as we shall see. If the discriminant of the touching conic (54) vanishes, the intersection of the corresponding planes will evidently give a focal line. Also the focal line is on the plane z, so that it satisfies

$$r\sin a+s\sin b+t\sin c=0 \ldots\ldots\ldots\ldots(56).$$

Thus we have, it is easy to see,

$$\frac{l^2}{\sin a}r+\frac{m^2}{\sin b}s+\frac{n^2}{\sin c}t=0 \ldots\ldots\ldots\ldots(57),$$

which with the similar equation for the other quartic and with the relation (56) gives the condition (55).

It is to be observed that to be given the four focal lines is only equivalent to six conditions on account of their being coplanar, and as the quartic cone involves 7 constants, viz. 12 minus the five conditions making a covariant cone of the second order stand on the imaginary circle at infinity, we have a singly infinite system of confocal and orthogonal cones.

We can also obtain the equation of the confocal cones in another form. Let V_1, V_2, V_3 denote three mutually orthogonal cones of the second order (42). Then from (50) it is easy to see that $V_1^2+V_2^2+V_3^2 \propto z^2(x^2+y^2+z^2)$, and that

$$\frac{V_1^2}{\lambda-a}+\frac{V_2^2}{\lambda-b}+\frac{V_3^2}{\lambda-c}=0 \ldots\ldots\ldots\ldots(58)$$

represents a confocal and orthogonal system of quartic cones, λ being a parameter and a, b, c being constants. If the

quartic has double contact with the cone standing on the imaginary circle at infinity, two of the focal lines coincide. The confocal quartics can then be written

$$\left.\begin{aligned}\sin a \sin\rho_1 - \sin b \sin\rho_2 &= k \sin\rho_3\\ \sin a \sin\rho_1 - \sin b \sin\rho_2 &= k' \sin\rho_3\end{aligned}\right\} \ldots\ldots\ldots(59),$$

and these cut orthogonally, k, k' being the parameters of the variable system. In the general case the confocal system referred to three focal lines can be written

$$\sin a \sin\rho_1 \sqrt{(t+\alpha)} + \sin b \sin\rho_2 \sqrt{(t+\beta)} + \sin c \sin\rho_3 \sqrt{(t+\gamma)} = 0 \quad \ldots\ldots\ldots(60),$$

where α, β, γ are constants and t is a variable parameter.

If the quartic cone has quartic contact with the cone standing on the imaginary circle at infinity, it can be written

$$\delta^2 z^2 (x^2 + y^2 + z^2) - u^2 = 0 \ldots\ldots\ldots\ldots(61),$$

where $$u = \alpha(x^2 + z^2) + \beta(y^2 + z^2) + 2\gamma xy,$$

and it has evidently then two nodal lines given by $z = 0$, $u = 0$. In this case it is easy to see that the cones (61), when α, β, γ are constants and δ is variable, are cut orthogonally by cones of the same form as $u = 0$, u' say, where α', β', γ' are such that $\alpha'\beta + \beta'\alpha - 2\gamma\gamma' = 0$, so that $u' = 0$ passes through four fixed points. It may be observed that (61) stands on the quadri-quadric curve $x^2 + y^2 + z^2 = r^2$, $u - r\delta z = 0$.

Returning now to the consideration of plane curves. I observe that the bicuspidal quartic, which involves 10 constants, has a centre. This can readily be seen by projecting the curve into a Cartesian oval, which has the circular points for cusps, and possesses an axis of symmetry. This quartic can be written in the unique canonical form

$$(z^2 + ax^2 + by^2 + 2hxy)^2 - x^2 y = 0 \ldots\ldots\ldots(62),$$

where $x = 0$, $z^2 + by^2 = 0$ give the cusps and y is the double tangent. This curve, then, has no conic related to it in the manner that we have been discussing. If, however, an invariant condition be satisfied, there are an infinite number of conics related to the curve, so that an infinite number of polygons of a certain number of sides can be circumscribed about one of them and inscribed in the curve. Suppose $ab - h^2 = 0$ in (62), then we can write

$$z^2 + ax^2 + by^2 + 2hxy = uv,$$

and the curve becomes

$$u^2 v^2 = x^2 y \ldots\ldots\ldots\ldots\ldots\ldots(63).$$

Now, describing a conic to touch the four lines u, v, x, y, which leaves an indeterminate parameter, the quartic (63) can be written in the form

$$\phi(\mu_1)\,\phi(\mu_2) = \text{a constant},$$

in μ_1, μ_2 coordinates, where

$$\phi(\mu) = \frac{(\mu-u)^2(\mu-v)^2}{\mu} \quad \ldots\ldots\ldots\ldots\ldots(64),$$

and u, v are the parameters of u, v, and ∞, 0 of x, y. Hence, according to the method we have used above, we see that the sixteen points, where a variable set of four tangents of the conic meet another connected variable of four tangents, lie on the curve. It may be observed that for the Cartesian oval the conic is a parabola touching the double tangent and having the triple focus of the curve as its focus.

I now consider the tri-central binodal quartic, and proceed to show that there are two conics related to it so as to touch the sides of inscribed polygons. This curve involves 10 constants, and can be projected into a central bicircular quartic, which can be written

$$(x^2+y^2+k^2)^2 - 4(a^2x^2+b^2y^2) = 0 \quad \ldots\ldots\ldots(65)$$

in rectangular coordinates. Now this curve is evidently the locus of the intersection of tangents to a conic which cut at a given angle θ, viz.

$$(x^2+y^2-a^2-b^2)^2 - 4a^2b^2\cot^2\theta\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right) = 0 \ldots(66),$$

where $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ is the conic. Now, to express the quartic (65) in the form (66), we have to find a concentric conic, having quartic contact with the curve, whose imaginary points of contact lie on its director circle. Such a conic with regard to (65) is

$$u = \theta^2 + \theta(x^2+y^2+k^2) + a^2x^2+b^2y^2 = 0 \ldots\ldots(67),$$

for the curve may be written $(x^2+y^2+k^2+2\theta)^2 = 4u$. The circle $x^2+y^2+k^2+2\theta=0$ thus is to be the director circle of u, which gives

$$(a^2+b^2-k^2)\theta^2 + 2a^2b^2\theta + a^2b^2k^2 = 0 \quad \ldots\ldots(68);$$

that is, there are two conics such as u.

Now it is evident that corresponding to $\theta = \frac{2\pi}{n}$ we shall have an infinite number of equiangular polygons, the exterior

angle being $\frac{2\pi}{n}$, with n vertices lying on the quartic, and whose sides touch u. Also it is evident that without imposing any restriction upon the angle we get an infinite number of quadrilaterals by considering two pairs of parallel tangents, two vertices lying on the inner and two vertices on the outer oval of the quartic, as the two ovals correspond to supplementary angles. I may observe that in the case of the equilateral triangles, the locus of the centres is found to be

$$(3x^2 + 3y^2 - a^2 - b^2)^2 = 4(a^2x^2 + b^2y^2 + a^2b^2) \quad \ldots\ldots(69),$$

as can be found from the invariants $x^2 + y^2 = a^2 + b^2 + r^2$,

$$R^4 - 2R^2(x^2 + y^2 + a^2 + b^2) + (x^2 + y^2)^2 - 2c^2(x^2 - y^2) + c^4 = 0,$$

and observing that

$$R^2 + 2r^2 = 0.$$

Further, it may be noticed that by taking different angles we can have an infinite number of polygons of n sides circumscribed to the conic so that each vertex moves along a quartic of the system (66) provided a certain relation connects the angles. These polygons, it may be observed, have their vertices at the intersection of alternate sides of certain other polygons with an even number of sides. These latter polygons have their vertices on equilateral hyperbolæ passing through the real and imaginary foci. Thus, let ω be the angle which a tangent at the conic makes with the axis, then if $\omega_1 + \omega_2 = \alpha$ the intersection of the tangents will lie on an equilateral hyperbola passing through the foci. Hence, for instance, considering a quadrilateral, we shall have an infinite number of such figures corresponding to

$$\left.\begin{array}{ll} \omega_1 + \omega_2 = \alpha, & \omega_2 + \omega_3 = \beta \\ \omega_3 + \omega_4 = \gamma, & \omega_4 + \omega_1 = \delta \end{array}\right\} \quad \ldots\ldots\ldots\ldots(70),$$

provided $\alpha + \gamma = \beta + \delta$, and the intersection of opposite sides will lie on the quartics $\omega_1 - \omega_2 = \alpha - \beta$, and $\omega_3 - \omega_4 = \beta - \gamma$, respectively. Projecting this result, we see that we can have an infinite number of polygons with an even number, $2n$, of sides circumscribed to a conic u, which have their vertices on the conics $V - k_1W$, $V - k_2W$, ..., $V - k_{2n}W$, where V, W intersect at the four vertices of a given quadrilateral circumscribed about u, provided the constants k_1, k_2, &c. are connected by a certain relation. And in the same case the intersections of alternate sides will lie on binodal tricentral quartics having quartic contact with u.

DETERMINATION OF SUCCESSIVE HIGH PRIMES.

[THIRD PAPER].

[*Showing* 213 *new High Primes.*]

By *Lt.-Col. Allan Cunningham, R.E.*, and *H. J. Woodall, Esq., A.R.C.Sc.*

22. *Scope of Paper.* This Paper is *in continuation* of *two* previous Papers with same Title (Vol. XXXI., pp. 165–176, and XXXIV., pp. 72–89 of this Journal), and the numbering of the Articles, Examples, and Tables in the present Paper is *in continuation* of the numbering* of those Papers. The method used for the determination of the *whole* of the High Primes (> 9 millions) within a given "Range"—say from $(N_0 - b)$ to $(N_0 + b')$—is fully explained therein, see Art. 1–5.

23. *Example* 7°. $N = (2^{26} \mp R)$. The number here chosen as "Central Number" is $N_0 = 2^{26}$, and the Range chosen is the group of 2160 numbers from $(2^{26} - 1080)$ to $(2^{26} + 1080)$. This work has been rendered possible by the help of the *Binary Canon*,† and the *Extension*‡ of the same, mentioned in Art. 5, 9.

23*a*. *Least Factor-Tables*§ (Tab. IX., X., XI.). These Tables show the *least* factors of *all* the numbers within the Range $(2^{26} \mp 1080)$ which are beyond the power of the large Factor-Tables. The numbers near 2^{26} being all > 67 millions, it has been found convenient to prepare *three* Least Factor-Tables (Tab. IX., X., XI.).

Tab. IX. All numbers (N) which are divisible by 2, 3, or 5 are *excluded* from this Table for the following reasons :–

(1) The *even* numbers are all such that their *quarters* or *halves* are of the form $(2^{24} \mp R_1)$ or $(2^{25} \mp R_2)$, already worked out under Ex. 1°, 2° (Art. 6, 7).

(2) The *odd* numbers (N), which contain the factor 3 or 5, are more conveniently treated separately, because $\frac{1}{3}N > 22$ million, and $\frac{1}{5}N > 13$ million (see under Tab. X., XI. below).

---

* Thus for Art. 1–12, Ex. 1°—3°, and Tab. I—IV, see the "first" Paper in Vol. XXXI. For Art. 13—21, Ex. 4°—6°, and Tab. V and VIII see the "second" Paper in Vol. XXXIV. Most of the references in the present Paper are to the first of these Papers (in Vol. XXXI).

† The *Binary Canon*, by Lt.-Col. Allan Cunningham, R.E., London, 1900, gives the solutions of $2^x \equiv R \pmod{p \text{ and } p^2}$ for all primes (p) and powers (p^2) of primes $\not> 1000$.

‡ The *Binary Canon Extension*, computed by the present authors (at present only in MS.), gives the *Residues* ($\pm R$), both + and − of 2^x, up to $x = 70$, to all prime, and powers of prime, moduli $\not> 10000$.

§ These Tables were styled in the first Paper (Art. 6*a*, &c.), *Factorisation-Tables*; but the title now used, *Least Factor-Tables*, is more appropriate.

Thus Tab. IX. contains only those numbers (within the Range) not containing the factors 2, 3, or 5; their general forms are

$N = N_0 - R = 2^{36} - (30m + r)$, where $r = 27, 23, 21, 17, 15, 11, 5, 3$;

$N = N_0 + R' = 2^{36} + (30m + r')$, where $r' = 3, 7, 9, 13, 15, 19, 25, 27$;

all other values of r, r' (< 30) giving numbers (N) divisible by 2, 3, or 5.

The Table is in *two* parts; the left half showing $(2^{36} - R)$, the right half showing $(2^{36} + R')$: part of the Argument $(30m)$ is in the *left* column of each half-Table, and the rest of the Argument (r, r') is in the Head-line. The body of the Table shows, for each *composite* number (N), at least one, viz., the *lowest* prime factor, or power of that factor (if repeated), followed (when space admits) by the next higher factor; also, when the width of the column (5 figures) admits, the residual factor is indicated (*if prime*) by the letter q or Q, viz., by q if < 9 million, and by Q if > 9 million (so that Q denotes a *High Prime*).

Use of $\sigma = 7$. When N contains the "Least Factor" 7, followed by a 4-figure factor, the symbol σ is used for 7, so as to include the *two* factors within the (5-figure) width of the column (the multiplication-symbol (.) between the two factors being then *omitted*).

23*b*. *Least Factor-Tables* (Tab. X., XI.). These Tables contain *all* the *odd* numbers (within the Range) which are divisible by 3 or 5 respectively (and thereby excluded from Tab. IX.), whose quotients $\frac{1}{3}N$, $\frac{1}{5}N$ are beyond the powers of the large Factor-Tables. Thus they show only those numbers (N) which are

(Tab. X.) divisible by 3, but not by 2, 9, 5; $\frac{1}{3}N > 22$ million.

(Tab. XI.) divisible by 5, but not by 2, 3, 25; $\frac{1}{5}N > 13$ million.

To save space the common factor 3 (in Tab. X.) or 5 (in Tab. XI.) is cast out (and placed at the head of the Table), so that the Tables contain really the numbers $\frac{1}{3}N$ or $\frac{1}{5}N$, whose general forms are

(Tab. X.) $\frac{1}{3}N = \frac{1}{3}(N_0 - R) = \frac{1}{3}\{2^{36} - (90m + r)\}$,
where $r = 73, 61, 55, 43, 37, 25, 7, 1$;

$\frac{1}{3}N = \frac{1}{3}(N_0 + R') = \frac{1}{3}\{2^{36} + (90m + r')\}$,
where $r' = 17, 29, 35, 47, 53, 65, 83, 89$;

(Tab. XI.) $\frac{1}{5}N = \frac{1}{5}(N_0 - R) = \frac{1}{5}\{2^{29} - (150m + r)\}$,
where $r = 149, 129, 119, 99, 69, 59, 29, 9$;

$\frac{1}{5}N = \frac{1}{5}(N_0 + R') = \frac{1}{5}\{2^{36} + (150m + r')\}$,
where $r' = 1, 21, 31, 51, 81, 91, 121, 141$;

all other values of r, r' giving numbers (N), which would be divisible by 2, 9, 5 (in Tab. X.), or by 2, 3, 25 (in Tab. XI.), and would therefore be within the powers of the large Factor-Tables. The arrangement and detail are quite similar to those of Tab. IX.

23*c*. *Use of Least Factor-Tables* (Tab. IX, X, XI). These Tables give in many cases sufficient data for the *complete* factorisation of N *without further reference* (as explained in Art. 6*c*, *q.v.*); and *all* the composite numbers are in any case brought within the power of the large Factor-Tables.

23*d*. *High Primes*. In all the Tables blanks indicate the absence of factors $< \sqrt{N}$, and therefore show High Primes $> 67, 22, 13$ million in Tab. IX., X., XI. respectively. The letter Q in Tab. IX. (in which it occurs only after the "Least Factor" 7) also indicates a High Prime, $\frac{1}{7}N > 9$ million. A separate List of all these High Primes is given in the Table below; the number of each kind is shown in the Abstract below

$p = (2^{36} \mp R)$; $p = \frac{1}{3}(2^{36} \mp R)$; $p = \frac{1}{5}(2^{36} \mp R)$; $p = \frac{1}{7}(2^{36} \mp R)$.

Number; 123 ; 42 ; 28 ; 20 .

List of 213 *High Primes* $p = \frac{1}{\mu} . N$ *between* $N = (2^{36} \mp 1080)$; $[\mu = 1, 3, 5, 7]$.

$\mu = 1$								$\mu = 3$			$\mu = 5$		$\mu = 7$	
67,107,...	67,108,				67,109,...			22,369,...			13,421,...		9,586,...	9,587,...
	003	303	661	933		243	629		481	801		797		003
	007	313	667	957		269	633		489	829		813		041
	009	331	669	961		281	663		499	871		819		063
	033	351	693	981		291	671		511	889		833		101
	037	369	709		023	299	687		513	891		839		111
787	039	373	721		033	321	701		531	903	563	843		119
793	049	387	729		051	323	723		549	931	567	869	847	129
797	081	439	739		071	363	737	273	573	937	579	873	859	
809	109	453	747		123	407	747	309	583	951	591	887	897	
863	127	463	753		131	417	759	343	597	967	651	897	903	
871	177	471	757		167	431	773	351	601	979	671	917	909	
881	183	493	763		171	443	923	381	661		677	927	933	
883	187	507	777		177	479	927	387	663		689	951	937	
913	199	511	819		191	507	941	393	667		711	981	939	
919	201	529	837		201	519		397	681		717		957	
941	207	579	859		209	543		421	703		731		961	
967	219	597	879		221	563		427	727		747		981	
977	271	633	913		227	573		447	733		767		987	
983	289	649	919		233	593		453	747		783		999	

24. *Tests of Work.* Same as in Art. 9, *q.v.*; also add—

Ex. 7°. The whole of the numerical work has been worked out by *two** computers *independently;* the results were then collated, and all discrepancies were further examined by both.

25. *Sequences of composites.* Add to previous Art. 11, 19 the following *exceptionally long* sequence of

149 composites between the primes 67109773 and 67109923.

The only other long sequences of composites at present known, nearly as long as, or longer than the above, are the four† named below

Next higher prime	2010881;	7230479;	8421403;	4652507;
Number of composites	147 ;	147 ;	151 ;	153

so that the one now reported is the *longest but two* at present known.

26. *Distribution of primes.* Add to previous Art. 12, 20 the following numbers (n) of primes within the Range of 1000 numbers from ($N_m - 1000$) to N_m.

$N_m = 2^{24}$, $n = 58$; $N_m = 2^{25}$, $n = 60$; $N_m = 2^{26}$, $n = 63$.

27. *Prime-pairs.* Two primes p, q, whose difference $p - q = 2$ are sometimes styled a *Prime-pair.* The following Abstract shows the number (n') of prime-pairs in the "Ranges" from ($N_0 - R$) to ($N_0 + R$) named below, and also the total number (n) of primes in the same Ranges (these cases are taken from Art. 6*b*, 7*b*, 7*d*, 8*b*, 23*d* of these Papers).

$N_0 =$	2^{26} ,	$\frac{1}{3}.2^{26}$,	$\frac{1}{5}.2^{26}$,	$\frac{1}{7}.2^{26}$;	2^{25} ,	$\frac{1}{3}.2^{25}$;	2^{24} ;	3^{15} ,
$R =$	1080,	360 ,	210 ,	154;	1020,	360;	1020;	1020,
$n' =$	6 ,	3 ,	0 ,	1 ;	13 ,	3 ;	8 ;	9 ,
$n =$	123 ,	42 ,	28 ,	20 ;	128 ,	45 ;	117 ;	127 .

Comparing these with those reported in Art. 21 tends to confirm Dr. Glaisher's statements‡ that (1) the percentage

* viz., by Miss A. Woodward (under Col. Cunningham's superintendence), and by Mr. Woodall.

† The Introduction to Glaisher's *Factor-Table for the 6th million*, pp. 64, 65, gives a list of *all* the long sequences of more than 100 composites occurring under 9 million. Three misprints occur in Art. 11 of the present authors' first Paper in quoting Glaisher's figures, (see the *Corrigenda* at end of this Paper).

‡ See pp. 28, 31 of Dr. Glaisher's Paper, *An Enumeration of Prime-pairs*, in Vol. VIII. of this Journal.

of prime-pairs (in a large Range of numbers) decreases on the whole as the numbers increase in magnitude, and (2) that n' is generally $< \frac{1}{10}n$.

28. *Summary of Results*, (1466 High Primes). The *easier cases* of factorisation by the method developed in Art. 1—5, (workable with the means at present* available), having now been worked out, so that any further work of the same kind would involve a greater degree of labour,† it seems worth while exhibiting here a brief Abstract of the *numbers* of High Primes thus detected. In this Table

N_0 is the "Central Number" used in the computations (Art. 5),

N is the general formula of the numbers in each Table,

N_0' is the (approximate) value of the middle number of each "Range,"

n is the number of High Primes in each "Range."

The High Primes thus found may be classed as follows—

7-fig. primes, 160; 8-fig. primes, 1196; 9-fig. primes, 110; Total, 1466.

It was not, of course, to be expected that the whole of the (1466) Higher Primes here reported would be *new* (*i.e.*, previously unpublished by other workers). Acknowledgments have been made of previous publication (by other workers) in all cases known to the present authors: these will all be found in the joint authors' Second Paper (Vol. XXXIV of this Journal, pp. 72—79).‡

The only important cases of *prior* publication were those by Mr. W. B. Davis (see Art. 17*d*, 17*e*) of

91 between 10^8 and $(10^8 + 1700)$; 11 of form $p = \frac{1}{11}(10^8 + R)$.

The work done in repeating Mr. Davis's work has been justified by confirming the 91 and 11 High Primes stated and detecting 8 mistakes (*i.e.*, 8 composite numbers reported as High Primes in his work).

---

* *i.e.*, with Binary and Denary Canons extending to moduli $\gtrless$ 10000.

† *i.e.*, either (using the existing Canons) the "Ranges" must be extended beyond ± 1020 (as was done in Ex. 6°, Art. 17); or new "Central Numbers" (N_0) of modified forms must be used (as was done in Ex. 4°, Art. 15); or else the present Canons (of Residues 2^n, 10^n) must be extended to moduli > 10000; or else new Canons must be formed with other Base-numbers.

‡ No case has been found of prior publication of the High Primes reported in these Papers arising from the use of the "Central Numbers," $N_0 = 3^{15}$, 2^{24}, 2^{25}, 2^{26}.

Number (n) of High Primes in the Ranges (R) near the Central Numbers (N_0').

N_0	N	N'_0	Range	n	N_0	N	N'_0	Range	n
2^{24}	$(2^{24} \mp R)$	16777216	− 1020 to + 1020	117	9.10^6	$(9.10^6 + R)$	9000510	0 to 1020	61
2^{25}	$(2^{25} \mp R)$	33554432	− 1020 to + 1020	128	10^7	$(10^7 \mp R)$	10000000	− 1050 to + 1020	118
,,	$\frac{1}{3}(2^{25} \mp R)$	11184811	− 1080 to + 1080	45	10^8	$(10^8 \mp R)$	100000000	− 2000 to + 2000	209
2^{26}	$(2^{26} \mp R)$	67108864	− 1080 to + 1080	123	,,	$\frac{1}{2}(10^8 \mp R)$	50000000	,, ,,	122
,,	$\frac{1}{3}(2^{26} \mp R)$	22369621	− 1080 to + 1080	42	,,	$\frac{1}{3}(10^8 \mp R)$	33333333	,, ,,	75
,,	$\frac{1}{5}(2^{26} \mp R)$	13421773	− 1050 to + 1050	28	,,	$\frac{1}{4}(10^8 \mp R)$	25000000	,, ,,	57
,,	$\frac{1}{7}(2^{26} \mp R)$	9586980	− 1080 to + 1080	20	,,	$\frac{1}{5}(10^8 \mp R)$	20000000	,, ,,	48
					,,	$\frac{1}{6}(10^8 \mp R)$	16666667	,, ,,	41
3^{15}	$(3^{15} \mp R)$	14348907	− 1020 to + 1020	127	,,	$\frac{1}{7}(10^8 \mp R)$	14285714	,, ,,	29
					,,	$\frac{1}{8}(10^8 \mp R)$	12500000	,, ,,	25
					,,	$\frac{1}{9}(10^8 \mp R)$	11111111	,, ,,	28
					,,	$\frac{1}{11}(10^8 \mp R)$	9090909	,, ,,	23

Least Factors of $2^{36}-(30m+r)$. Table IX. *Least Factors of* $2^{36}+(30m+r')$.

30m	r 27	23	21	17	15	11	5	3
1050		19q			11q	49.67		6733
1020	7.811	11.43	17.17	29q	19q	13q	31.59	293q
990	691q	71	197q	17.41	7.13		47.61	
960	3517			7.11q	2027	79q	37	49.31
930	43.83	13.53		211		157q	7Q	121q
900	13.67		7.19	659q	139	11q	17	149
870		σ1039	37q		7883		13	61q
840	11.23	29				7Q	11.19	41q
810	7.17q	593q				23		79
780	19.47	17q	11q	13q	7.853	83	43q	
750	31	5881	13	49.73	607q	41		7.383
720	29q	37	53q		11.17	19q	7.109	3733
690	8011	11.47	7.107	653q	2579	17	5953	13.19
660		7.23q			97q	59q		
630		31	641q	11		7.13	53	17
600	7.157	163q	73	19	13	1063	149q	11
570	41q		23.31	71	7Q	11		29q
540	419q	13q		σ2969	2539		23q	7Q
510	11.13		17q	43q	47.67	37q	7.11	
480	61	19q	7Q	17.79			13.13	1877
450		499	11	31q	19q	89q	1303	23.23
420	37q	2797	43	233	59	7.47q		53q
390	7.521	127q		13.23	11.31		17q	
360	151	11q	13	1237	49q		5231	1091
330			19.41	7.103	367q	29q		7Q
300	17q	101q	271q	11.53	23	113	7Q	13.71
270	163q	17.83	7Q	97q		31	19	11q
240		7.61q	173q	67q	193	11.13	241q	47q
210	19q	1399		4129	13.17	7.727		2903
180	7.121		421q			17q	11	43.73
150	23q	13		29q	7Q	19.71		179q
120	13.13		11.61	7Q		23q		7.17
90		503q			41.53		7.13	97q
60		647q	49.59	37q	11q	109q	6029	1699
30	757q	7.11	29q	19.19		1433	3623	31q
·		41.79	17q	13q	3911	7.43q		37

30m	r' 3	7	9	13	15	19	25	27
·	7Q	23.29	13q	11.17		883q	1129	5437
30	7681	59q	47.83	6163	7Q			11
60	3677	19q		7.89q	43q	11q	173q	7.13
90			23q	61q	19	73q	343	
120	11.29	5657	7Q	47q	79q	13q	11.23	109q
150	31q	7Q		3067	13q		137q	373q
180	17q		11q	6871	571q	σ2251	83	
210	49.53	13 17	19.37	41	61q	839q	857q	23
240	13	43q	389q	167	7.11		71	
270	59	121	67q	7.23	17q	149	13.19	7.29q
300			4547		157q	17.53	7.181	
330	19.43		7.31q	11		127q	29	
360		7.37		13	23		6763	11.17
390	331q	1409	13	643q		49.11	47	
420	7Q		89q	941q		719q	1021	19
450	11q			31q	7.739	41.61	11	13
480	661q	3019	17.97	σ1123	233q		43q	49
510	23q	193q	121	17.19	31	13.13	σ2389	83
540		67.73	7.71q		13	23q	197q	
570	107q	7Q		109	11q	37	6857	3359
600	47	11.13	79.79	3463		σ1301	17	59
630	7.13	19.53	1051		29q	31.31		1861
660	37	683q	887q	11q	7.19		13	101
690	17.17	23.47		49.29	191q		41q	7.11q
720	137q	17q		131q	227	11q	7.59q	6053
750	139q	277q	σ1459	13.43			257q	173q
780	11q	7.107	13.19	3331	17q		11q	
810	4447	5477	29		2851	7.17q	23.31	
840	7Q	1453	11.43	431q	3659		19q	13.13
870		29.97	41q		7.37	101		17.31
900	19	283q		7Q	11q	13q	79	7.23
930	1487	11	3571	157q	13q	67q	7.53	61q
960	89q	2423	7Q	23	71q	19.47	73q	269
990	29q	49.13	17q	121q	1979	8069	2683	19
1020	13q	389q	307q	17.37	83q	7Q	113	11q
1050	σ1949	3191			23q	11q	13q	

Least Factors of $\frac{1}{3}\{2^{36}-(90m+r)\}$. Table X. *Least Factors of* $\frac{1}{3}\{2^{36}+(90m+r')\}$.

90m	r								90m	r'							
	73	61	55	43	37	25	7	1		17	29	35	47	53	65	83	89
990	97q	23q		7q	193q	2657	19q	7.151	.	49q	19.31	121	17q	23	29	359q	4673
900	17	2741	11q	109q		41	7.59q	13.31	90	53q				7.19q	59q	563q	
810	19	17 43	49q	2083	53q		37.89		180	139q	131q	31	7.47q	11q		1723	7.13q
720	67q	7q	23q	113q	11q	13	101		270	379q	11q	41		191q		7.17	107q
630		121			13 17	7.19	23	29q	360		367q	7q	71q	29.29	13.53	953q	137
540	7.43q		37q		131q	17q	2243	19	450	23q	7q	19	11	13	37q	223q	
450		13		11q	7q	73q	29	881q	540	17q	2593	811q	29.31	113q	49.23		11
360	13.41		167q	7q		61q		7.11	630	7q	13.17	991q	43	97	11	19.61	241q
270	31q			19	149q	11	49.13		720	13.37		251q	79q	7.31q	89q		
180	257q	2999	7q	23		3187	41q	79q	810	11.19	73q		7.127	17.59	3967	11.13	49q
90	11	7.37q		1499	83q		11.43	4001	900	1609		29.43		3209	17	7q	
.			17.47	13	73	7	1493	2731	990	163q	23	7.11		41q	19		61q

Least Factors of $\frac{1}{5}\{2^{36}-(150m+r)\}$. Table XI. *Least Factors of* $\frac{1}{5}\{2^{36}+(150m+r')\}$.

150m	r								150m	r'							
	149	129	119	99	69	59	29	9		1	21	31	51	81	91	121	141
900			7q	11		19.23	101q		.	53	619q	7q		17	31q		41
750	1231	7.577	97	13	2143	7q	11q	103q	150	2843	7.197	19.29			7.509	103	947q
600	167q	23.71	13q	43	49.11	67q	61.61		300		11.13			7q	59q	17.23	37q
450	7.17q	19q	373q	2887	541q			7.13	450	7.13	29			53q	11q		7.17
300	121q	17q		29	337q	139	13.73		600	2531		641q	11.23	67	71q		211q
150	37q		23.41	7.251	31q		49q	127q	750	19q		2141	343	1381	13q	7.11q	
.	47q		11	137	13	191q		11.19	900	199q	167	17q	733q	11.59	2161	31.43	

Addendum to Second Paper, (Vol. XXXIV.)

p. 78; Art. 17*d*, i—iii. Dr. T. B. Sprague has recently effected a *partial* factorisation of all the *odd* numbers from 10^8 to $(10^8 + 2000)$ to the extent of casting out the *simple* prime factors < 10000, (but not the repeated factors). His work confirms the present author's determination of the same, and thereby also confirms (as being *complete*) their determination of the High Primes $> 10^8$ within that Range.

Corrigenda on First Paper, (Vol. XXXI.)

p. 171, line 12 up; For 1170, Read 1700.

p. 171, At foot; For ....1169, Read ....1699.

p. 172, Table of Art. 11; Read 113 (not 131) in 1st million;
Read 137 (not 157) in 4th million;
Read 131 (not 151) in 6th million;

In last line of same Table; Read 1700 (not 1170)

p. 172, line 7 up; For ‡58, Read ‡54.

Corrigendum on Second Paper, (Vol. XXXIV.)

p. 74, Art. 17, line 1. For $(10^8 \mp N)$, Read $(10^8 \mp R)$.

END OF VOL. XXXIV.

METCALFE AND CO, LIMITED, CAMBRIDGE.

CONTENTS.

PAGE

On the plane quartic curve with a centre and the corresponding cone (continued). By R. A. ROBERTS - - - - - - - - - - 177

Determination of successive high primes. [Third paper.] By LT.-COL. CUNNINGHAM and H. J. WOODALL - - - - - - - - 184

Papers for insertion should be sent to Dr. Glaisher, Trinity College, Cambridge, or to Messrs. Metcalfe and Co. Limited, Printing Office, Trinity Street, Cambridge.

An index to vols. i—xxv has been published. It is issued gratis to subscribers, but may be obtained separately, price Two Shillings.

To avoid fine, this book should be returned on
or before the date last stamped below

MAY 1 1 '56

510.5
M584
v.34

LIBRARY
STORAGE AREA